Kisin modules in all characteristics
(except 2)

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Outline

Let k be a finite field, characteristic $p > 2$.
Let R be a discrete valuation ring, char $R = 0$, residue field k.

Objectives.

1. Construct (finite, commutative, cocommutative p-power rank) Hopf algebras over R.
2. Construct (finite, commutative, cocommutative p-power rank) Hopf algebras over $k[[t]]$.
3. Find relationships between these constructions.

Key tool. Kisin modules (née Breuil-Kisin modules).

Outline

Let:

- $W = W(k)$ ring of Witt vectors, $W_n = W/p^n W$ length n vectors
- $\mathfrak{S} = W[[u]], \mathfrak{S}_n = \mathfrak{S}/p^n\mathfrak{S} = W_n[[u]]$
- $\sigma : \mathfrak{S} \to \mathfrak{S}$ be Frobenius-semilinear map, $u \mapsto u^p$
  - Write $\sigma(f) = f^\sigma$.
  - $(pf)^\sigma \in p\mathfrak{S}$
  - We also have $\sigma : \mathfrak{S}_n \to \mathfrak{S}_n$
- for M an $\mathfrak{S}$-module, $M^\sigma = \mathfrak{S} \otimes_{\mathfrak{S}} M$ with

$$s_1 \otimes_\sigma s_2 m = s_1 s_2^\sigma \otimes_\sigma m; s_1, s_2 \in \mathfrak{S}, m \in M$$

- for D a complete dvr with residue field k, pick $E \in \mathfrak{S}$ such that $E(0) = p$ and $D \cong \mathfrak{S}/E\mathfrak{S}$.
  - $\mathrm{char}(D) = 0 \Rightarrow E$ is an Eisenstein polynomial.
  - $\mathrm{char}(D) = p \Rightarrow E = p$

If $p^n M = 0$ then we may assume $M^\sigma = \mathfrak{S}_n \otimes_{\mathfrak{S}_n} M$.

Definition

A Kisin module relative to $\mathfrak{S} \to D$ is a triple (M, φ, ψ) where

- M is a $\mathfrak{S}$-module which:
  - is finitely generated
  - is killed by a power of p
  - has projective dimension at most 1.
- $\varphi : M \to M^\sigma$ and $\psi : M^\sigma \to M$ are $\mathfrak{S}$-linear maps with

$$\varphi\psi = E \text{ and } \psi\varphi = E$$

Remarks.

- $\varphi\psi \neq \psi\varphi$: $\varphi\psi \in \mathrm{End}(M^\sigma)$ and $\psi\varphi \in \mathrm{End}(M)$.
- The $\mathfrak{S}$-module M does not depend on D.
  - Alternatively, for a given M we say (φ, ψ) give a Kisin structure relative to $\mathfrak{S} \to D$.
- Write $M = (M, \varphi, \psi)$.

There is an equivalence:

$$\left\{ \begin{array}{c} \text{Kisin modules} \\ \text{relative to } \mathfrak{S} \to D \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{abelian } D\text{-Hopf algebras} \\ \text{of } p\text{-power rank} \end{array} \right\}$$

$$M \mapsto H_M$$

$$M_H \leftarrow\!\shortmid H$$

- M is a $\mathfrak{S}$-module which:
  - is finitely generated: required for H_M to have finite rank.
  - is killed by a power of p: $p^n M = 0 \leftrightarrow [p^n]H_M = 0$.
  - has proj. dim. $M \leq 1$: projective resolution for $M \leftrightarrow$ isogeny of formal groups with cokernel H_M.
- $\varphi : M \to M^\sigma$ and $\psi : M^\sigma \to M$ are $\mathfrak{S}$-linear maps with $\varphi\psi = E$ and $\psi\varphi = E$: φ and ψ analogous to F and V for Dieudonné modules.

Outline

Example

Let $M = \mathfrak{S}_n\mathbf{e} \cong \mathfrak{S}_n$ (as $\mathfrak{S}$-modules).
Let $\varphi(\mathbf{e}) = E \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = \mathbf{e}$.

- M is a $\mathfrak{S}$-module which:
  - is finitely generated: clear.
  - is killed by a power of p: $p^n M = 0$.
  - has proj. dim. $M \leq 1$: $\mathfrak{S} \to \mathfrak{S} \to M$ is a projective resolution.
- $\varphi : M \to M^\sigma$ and $\psi : M^\sigma \to M$ are $\mathfrak{S}$-linear maps with $\varphi\psi(1 \otimes_\sigma \mathbf{e}) = E \otimes_\sigma \mathbf{e}$ and $\psi\varphi(\mathbf{e}) = E\mathbf{e}$: clear.

Example

Let $M = \mathfrak{S}_1\mathbf{e} = k[[u]]\mathbf{e}$ (so $M^\sigma = \mathfrak{S}_1 \otimes_{\mathfrak{S}_1} M$).

$D = R$ (characteristic zero):

- $E\mathbf{e} = u^e\mathbf{e}, E \otimes_\sigma \mathbf{e} = u^e \otimes_\sigma \mathbf{e}$.
- $\varphi(\mathbf{e})$ is a factor of $u^e \otimes_\sigma \mathbf{e}$, say $\varphi(\mathbf{e}) = u^r f \otimes_\sigma \mathbf{e}, r \leq e, f \in \mathfrak{S}_1^\times$.
- $\psi(1 \otimes_\sigma \mathbf{e}) = u^{e-r} f^{-1}\mathbf{e}$.

$D = k[[t]]$ (characteristic p):

- $E\mathbf{e} = p\mathbf{e} = 0, E \otimes_\sigma \mathbf{e} = p \otimes_\sigma \mathbf{e} = 1 \otimes_\sigma p\mathbf{e} = 0$.
- Two cases:
  - **Case 1.** $\varphi(\mathbf{e}) = 0 \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = f\mathbf{e}, f \in \mathfrak{S}_1$.
  - **Case 2.** $\varphi(\mathbf{e}) = f \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = 0, f \in \mathfrak{S}_1$.

Example

Let $M = \mathfrak{S}_n \mathbf{e} = W_n[[u]]\mathbf{e}, n > 1$.

$D = R$:

- $\varphi(\mathbf{e})$ is a factor of $E \otimes_\sigma \mathbf{e}$, but E is irreducible.
    - **Case 1.** $\varphi(\mathbf{e}) = fE \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = f^{-1}\mathbf{e}, f \in \mathfrak{S}_n^\times$.
    - **Case 2.** $\varphi(\mathbf{e}) = f \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = f^{-1}E\mathbf{e}, f \in \mathfrak{S}_n^\times$.

$D = k[[t]]$:

- $E\mathbf{e} = p\mathbf{e}, E \otimes_\sigma \mathbf{e} = p \otimes_\sigma \mathbf{e}$.
- $\varphi(\mathbf{e})$ is a factor of $p \otimes_\sigma 1\mathbf{e}$.
    - **Case 1.** $\varphi(\mathbf{e}) = f \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = f^{-1}p, f \in \mathfrak{S}_n^\times$.
    - **Case 2.** $\varphi(\mathbf{e}) = fp \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = f^{-1}, f \in \mathfrak{S}_n^\times$.

Outline

In this section, the dvr R will vary (but always be characteristic 0). Let E_R be the Eisenstein polynomial for R with $E(0) = p$ and let $e_R = e(\mathrm{Frac}(R)/\mathbb{Q}_p)$.

- M is a $\mathfrak{S}$-module
- $\varphi : M \to M^\sigma$ and $\psi : M^\sigma \to M$ are $\mathfrak{S}$-linear maps with $\varphi\psi = E$ and $\psi\varphi = E$

Fact. In characteristic zero, φ is injective, hence $\psi(x) = \varphi^{-1}(Ex)$ and is uniquely determined.

Proposition

In characteristic zero, a K-module relative to $\mathfrak{S} \to R$ can be viewed as a pair $(M, \varphi), \varphi : M \to M$ a σ-semilinear map such that

- $M \cong \oplus_{i=1}^{c} \mathfrak{S}_{n_i}$.
- for all $m \in \mathfrak{M}$,

$$E_R m = \sum s_i \varphi(m_i), s_i \in W[[u]], m_i \in M$$

By allowing R to vary, we can construct families of K-modules.

- $M \cong \oplus_{i=1}^{c} \mathfrak{S}_{n_i}$.
- for all $m \in \mathfrak{M}$,

$$E_R m = \sum s_i \varphi(m_i), s_i \in W[[u]], m_i \in \mathfrak{M}$$

By allowing R to vary, we can construct families of K-modules.

Example

Let $M = \mathfrak{S}_1 \mathbf{e}_i, \varphi(\mathbf{e}) = u^r \mathbf{e}$ (or $\varphi(\mathbf{e}) = u^r \otimes_\sigma \mathbf{e}$).
For all R with $e_R \geq r$ we have

$$E_R \mathbf{e} = u^{e_R - r} u^r \mathbf{e} = u^{e_R - r} \varphi(\mathbf{e})$$

and (M, φ) is a K-module relative to $\mathfrak{S} \to R$.

Typically, K-modules relative to a family of dvr's are easy to construct when M is killed by p.

Example

Let $M = \mathfrak{S}_1\mathbf{e}_1 \oplus \mathfrak{S}_1\mathbf{e}_2$ and

$$\varphi(\mathbf{e}_1) = u\mathbf{e}_1 + u^9\mathbf{e}_2$$
$$\varphi(\mathbf{e}_2) = u^7\mathbf{e}_1 + u^6\mathbf{e}_2$$

Generally,

$$s_1\varphi(\mathbf{e}_1) + s_2\varphi(\mathbf{e}_2) = (us_1 + u^7 s_2)\mathbf{e}_1 + (u^9 s_1 + u^6 s_2)\mathbf{e}_2$$

hence

$$(1-u^9)^{-1}\varphi(\mathbf{e}_1) - u^3(1-u^9)^{-1}\varphi(\mathbf{e}_2) = u\mathbf{e}_1$$
$$-u^6(1-u^9)^{-1}\varphi(\mathbf{e}_1) + (1-u^9)^{-1}\varphi(\mathbf{e}_2) = u^6\mathbf{e}_2,$$

so (M, φ) is a K-module relative to $\mathfrak{S} \to R$ provided $e_R \geq 6$.

Cyclic case – conditions simplify to:

- $M = \mathfrak{S}_n \mathbf{e}$
- $E_R \mathbf{e} = s\varphi(\mathbf{e}), s \in \mathfrak{S}_n$.

Suppose $n \geq 2$.
We require a factorization of $E_R \in \mathfrak{S}_n$.
But E_R is irreducible in $\mathfrak{S}_n$, so it follows that $\varphi(\mathbf{e}) = f\mathbf{e}$ or $\varphi(\mathbf{e}) = E_R f\mathbf{e}, f \in \mathfrak{S}_n^\times$.
In either case, we can replace f with $b = f(0) \in W_n^\times$.
Thus $\varphi(\mathbf{e}) = b\mathbf{e}$ or $\varphi(\mathbf{e}) = bE\mathbf{e}$ for some Eisenstein polynomial E.
In the first case, (M, φ) is a K-module relative to $\mathfrak{S} \to R$ for every R.
In the second, (M, φ) is a K-module relative only to $\mathfrak{S} \to W[[u]]/(E)$.

Example

Let $M = \mathfrak{S}_n\mathbf{e}_1 + \mathfrak{S}_n\mathbf{e}_2, \varphi(\mathbf{e}_1) = u^3\mathbf{e}_1 + \mathbf{e}_2, \varphi(\mathbf{e}_2) = p\mathbf{e}_1 - \mathbf{e}_2$.
Then

$$\varphi(\mathbf{e}_1) + \varphi(\mathbf{e}_2) = u^3\mathbf{e}_1 + \mathbf{e}_2 + p\mathbf{e}_1 - \mathbf{e}_2 = (u^3 + p)\mathbf{e}_1$$
$$p\varphi(\mathbf{e}_1) - u^3\varphi(\mathbf{e}_2) = pu^3\mathbf{e}_1 + p\mathbf{e}_2 - pu^3\mathbf{e}_1 + u^3\mathbf{e}_2 = (u^3 + p)\mathbf{e}_2$$

So (M, φ) is a K-module relative to $\mathfrak{S} \to W[\sqrt[3]{-p}]$.

$E = u^3 + p$ is the only Eisenstein polynomial obtained as $\mathfrak{S}_n$-linear combinations of $\varphi(\mathbf{e}_1)$ and $\varphi(\mathbf{e}_2)$, $n \geq 2$.

Example

Let $M = \mathfrak{S}_n\mathbf{e}_1 + \mathfrak{S}_n\mathbf{e}_2, \varphi(\mathbf{e}_1) = \mathbf{e}_1 + u^2\mathbf{e}_2, \varphi(\mathbf{e}_2) = u\mathbf{e}_1 + \mathbf{e}_2$.
Then

$$s_1\varphi(\mathbf{e}_1) + s_2\varphi(\mathbf{e}_2) = (s_1 + s_2u)\mathbf{e}_1 + (s_1u^2 + s_2)\mathbf{e}_2.$$

Notice

$$\begin{aligned}(1-u^3)\mathbf{e}_1 &= \varphi(\mathbf{e}_1) - u^2\varphi(\mathbf{e}_2)\\ (1-u^3)\mathbf{e}_2 &= -u\varphi(\mathbf{e}_1) + \varphi(\mathbf{e}_2)\end{aligned}$$

so given E_R we have

$$\begin{aligned}E_R\mathbf{e}_1 &= E_R(1-u^3)^{-1}\varphi(\mathbf{e}_1) - u^2E_R(1-u^3)^{-1}\varphi(\mathbf{e}_2)\\ E_R\mathbf{e}_2 &= -uE_R(1-u^3)^{-1}\varphi(\mathbf{e}_1) + E_R(1-u^3)^{-1}\varphi(\mathbf{e}_2)\end{aligned}$$

What's the difference between these two examples?

Outline

Rough idea.

1. Start with a Kisin module relative to $\mathfrak{S} \to R$.
2. Use it to construct a K-module relative to $\mathfrak{S} \to R_1$ for R_1 an extension of R.
3. Repeat, obtaining a K-module relative to a tower of extensions.

Write $R = W[[u]]/(E)$, (recall $E(0) = p$), and let $E(\pi) = 0, \pi \in R$.
Then E^σ is an Eisenstein polynomial, $E^\sigma(0) = p$, and
$R_1 = W[[u]]/(E^\sigma)$ is an extension of R of degree p.
$R_1 = R[\pi_1], \pi_1^p = \pi$.
More generally, we have $R_m = W[[u]]/(E^{\sigma^m}) = R[\sqrt[p^m]{\pi}]$.
Write $E = wu^e + puF + p, w \in W, F \in \mathfrak{S}, \deg F < e - 1$.
Then

$$\lim_{m\to\infty} E^{\sigma^m} = w^{\sigma^m} u^{p^m e} - pu^{pm} F^{\sigma^m} + p^{\sigma^m} = p$$

in the u-adic topology, so

$$W[[u]]/(E^{\sigma^m}) \to W[[u]]/p = k[[u]].$$

Let (M, φ) be a K-module relative to $\mathfrak{S} \to R$.
Let $\mathbf{e}_1, \ldots, \mathbf{e}_c$ generate M, and let $\varphi(\mathbf{e}_i) = \sum_{j=1}^{c} f_j \mathbf{e}_j$.
For all i we can write

$$E\mathbf{e}_i = \sum_{j=1}^{c} s_{i,j}\varphi(\mathbf{e}_j), s_{i,j} \in \mathfrak{S}.$$

Define $\varphi_1 : M \to M$ to be the semilinear map $\varphi_1(\mathbf{e}_i) = \sum_{j=1}^{c} f_j^{\sigma} \mathbf{e}_j$. Then

$$E^{\sigma}\mathbf{e}_i = \sum_{j=1}^{n} s_{i,j}^{\sigma}\varphi_1(\mathbf{e}_j)$$

and so (M, φ_1) is a K-module relative to $\mathfrak{S} \to R_1$.
This is, or is close to, base change.

Example

Let $M = \mathfrak{S}_1\mathbf{e}, \varphi(\mathbf{e}) = 1 \otimes_\sigma \mathbf{e}$ using char-free def. of K-mod.

$$\varphi(\mathbf{e}) = 1 \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = u^e\mathbf{e} \text{ gives } \mathfrak{S} \to R$$
$$\varphi_1(\mathbf{e}) = 1 \otimes_\sigma \mathbf{e}, \psi_1(1 \otimes_\sigma \mathbf{e}) = u^{pe}\mathbf{e} \text{ gives } \mathfrak{S} \to R_1$$
$$\varphi_2(\mathbf{e}) = 1 \otimes_\sigma \mathbf{e}, \psi_2(1 \otimes_\sigma \mathbf{e}) = u^{p^2e}\mathbf{e} \text{ gives } \mathfrak{S} \to R_2$$
$$\vdots$$
$$\varphi_m(\mathbf{e}) = 1 \otimes_\sigma \mathbf{e}, \psi_m(1 \otimes_\sigma \mathbf{e}) = u^{p^me}\mathbf{e} \text{ gives } \mathfrak{S} \to R_m$$

Let $m \to \infty$. Then

$$\varphi_\infty(\mathbf{e}) = 1 \otimes_\sigma \mathbf{e}, \psi_\infty(1 \otimes_\sigma \mathbf{e}) = 0$$

gives a K-module structure relative to $\mathfrak{S} \to k[[t]]$.

Example

Let $M = \mathfrak{S}_1\mathbf{e}, \varphi(\mathbf{e}) = u^e \otimes_\sigma \mathbf{e}$.

$$\varphi(\mathbf{e}) = u^e \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = \mathbf{e} \text{ gives } \mathfrak{S} \to R$$
$$\varphi_1(\mathbf{e}) = u^{pe} \otimes_\sigma \mathbf{e}, \psi_1(1 \otimes_\sigma \mathbf{e}) = \mathbf{e} \text{ gives } \mathfrak{S} \to R_1$$
$$\varphi_2(\mathbf{e}) = u^{p^2e} \otimes_\sigma \mathbf{e}, \psi_2(1 \otimes_\sigma \mathbf{e}) = \mathbf{e} \text{ gives } \mathfrak{S} \to R_2$$
$$\vdots$$
$$\varphi_m(\mathbf{e}) = u^{p^me} \otimes_\sigma \mathbf{e}, \psi_m(1 \otimes_\sigma \mathbf{e}) = \mathbf{e} \text{ gives } \mathfrak{S} \to R_m$$

Let $m \to \infty$. Then

$$\varphi_\infty(\mathbf{e}) = 0 \otimes_\sigma \mathbf{e} = 0, \psi_\infty(1 \otimes_\sigma \mathbf{e}) = \mathbf{e}$$

gives a K-module structure relative to $\mathfrak{S} \to k[[t]]$.

Example

Let $M = \mathfrak{S}_1\mathbf{e}, \varphi(\mathbf{e}) = u^r \otimes_\sigma \mathbf{e}, 0 < r < e$. Then:

$$\varphi(\mathbf{e}) = u^r \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = u^{e-r}\mathbf{e} \text{ gives } \mathfrak{S} \to R$$
$$\varphi_1(\mathbf{e}) = u^{pr} \otimes_\sigma \mathbf{e}, \psi_1(1 \otimes_\sigma \mathbf{e}) = u^{p(e-r)}\mathbf{e} \text{ gives } \mathfrak{S} \to R_1$$
$$\varphi_2(\mathbf{e}) = u^{p^2 r} \otimes_\sigma \mathbf{e}, \psi_2(1 \otimes_\sigma \mathbf{e}) = u^{p^2(e-r)}\mathbf{e} \text{ gives } \mathfrak{S} \to R_2$$
$$\vdots$$
$$\varphi_m(\mathbf{e}) = u^{p^m r} \otimes_\sigma \mathbf{e}, \psi_m(1 \otimes_\sigma \mathbf{e}) = u^{p^m(e-r)}\mathbf{e} \text{ gives } \mathfrak{S} \to R_m$$

Let $m \to \infty$. Then

$$\varphi_\infty(\mathbf{e}) = 0 \otimes_\sigma \mathbf{e} = 0, \psi_\infty(1 \otimes_\sigma \mathbf{e}) = 0$$

gives a (trivial) K-module structure relative to $\mathfrak{S} \to k[[t]]$.

Example

Let $M = \mathfrak{S}_n\mathbf{e}, n \geq 2$. Either:

$$\varphi(\mathbf{e}) = fE \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = f^{-1}\mathbf{e} \text{ gives } \mathfrak{S} \to R$$

$$\varphi_m(\mathbf{e}) = f^{\sigma^m} E^{\sigma^m} \otimes_\sigma \mathbf{e}, \psi_m(1 \otimes_\sigma \mathbf{e}) = (f^{-1})^{\sigma^m}\mathbf{e} \text{ gives } \mathfrak{S} \to R_1$$

$\varphi_\infty(\mathbf{e}) = pb \otimes_\sigma \mathbf{e}, \psi_\infty(1 \otimes_\sigma \mathbf{e}) = b^{-1}$ if $f(0) = b \in W_n(\mathbb{F}_p)$.

Or:

$$\varphi(\mathbf{e}) = f \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = f^{-1}E\mathbf{e} \text{ gives } \mathfrak{S} \to R$$

$$\varphi_m(\mathbf{e}) = f^{\sigma^m} \otimes_\sigma \mathbf{e}, \psi_1(1 \otimes_\sigma \mathbf{e}) = (f^{-1})^{\sigma^m} E^{\sigma^m}\mathbf{e} \text{ gives } \mathfrak{S} \to R_1$$

$\varphi_\infty(\mathbf{e}) = b \otimes_\sigma \mathbf{e}, \psi_\infty(1 \otimes_\sigma \mathbf{e}) = pb^{-1}$ if $f(0) = b \in W_n(\mathbb{F}_p)$.

$$\varphi(\mathbf{e}) = fE \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = f^{-1}\mathbf{e} \text{ gives } \mathfrak{G} \to R$$

Taking the limit doesn't work well if $b \notin W_n(\mathbb{F}_p), b = f(0)$.
But, $\varphi_\infty(\mathbf{e}) = b \otimes_\sigma \mathbf{e}, \psi_\infty(1 \otimes_\sigma \mathbf{e}) = pb^{-1}$ if $b \notin W_n(\mathbb{F}_p)$ still works:

$$\psi_\infty\varphi_\infty(\mathbf{e}) = \psi_\infty(b \otimes_\sigma \mathbf{e}) = bpb^{-1}\mathbf{e} = p\mathbf{e}$$

$$\varphi_\infty\psi_\infty(1 \otimes_\sigma \mathbf{e}) = \varphi_\infty(pb^{-1}) = pb^{-1}b \otimes_\sigma \mathbf{e} = p \otimes_\sigma 1\mathbf{e}$$

Generally, suppose M is generated by $\{\mathbf{e}_1, \dots, \mathbf{e}_c\}$ and

$$\varphi(\mathbf{e}_i) = \sum_{j=1}^{c} f_{i,j} \otimes_\sigma \mathbf{e}_j, \psi(1 \otimes_\sigma \mathbf{e}_j) = \sum_{i=1}^{c} g_{j,i}\mathbf{e}_i, f_{i,j}, g_{j,i} \in \mathfrak{S}$$

is a K-module structure on M relative to $\mathfrak{S} \to R$. Then

$$E \otimes_\sigma \mathbf{e}_j = \varphi\psi(1 \otimes_\sigma \mathbf{e}_j) = \sum_{j=1}^{c}\sum_{i=1}^{c} f_{i,j} g_{j,i} \otimes_\sigma \mathbf{e}_j$$

$$1 \otimes_\sigma p\mathbf{e}_i = \sum_{j=1}^{c}\sum_{i=1}^{c} f_{i,j}(0) g_{j,i}(0) \otimes_\sigma \mathbf{e}_j$$

and we get a K-module structure on M relative to $\mathfrak{S} \to k[[t]]$:

$$\varphi_\infty(\mathbf{e}_i) = \sum_{j=1}^{c} f_{i,j}(0) \otimes_\sigma \mathbf{e}_j,$$

$$\psi_\infty(1 \otimes_\sigma \mathbf{e}_j) = \sum_{i=1}^{c} g_{j,i}(0)\mathbf{e}_i, f_{i,j}(0), g_{j,i}(0) \in W.$$

Example (Last one.)

Let $R = W[\sqrt[3]{-p}]$. Then $E = u^3 + p$. Let $M = \mathfrak{S}_n\mathbf{e}_1 + \mathfrak{S}_n\mathbf{e}_2$ and

$$\varphi(\mathbf{e}_1) = u^3 \otimes_\sigma \mathbf{e}_1 + 1 \otimes_\sigma \mathbf{e}_2 \quad \psi(1 \otimes_\sigma \mathbf{e}_1) = \mathbf{e}_1 + \mathbf{e}_2$$
$$\varphi(\mathbf{e}_2) = p \otimes_\sigma \mathbf{e}_1 - 1 \otimes_\sigma \mathbf{e}_2 \quad \psi(1 \otimes_\sigma \mathbf{e}_2) = p\mathbf{e}_1 - u^3\mathbf{e}_2$$

(Recall $(\psi(x) = \varphi^{-1}((u^3 + p)x))$.)

Then

$$\varphi_\infty(\mathbf{e}_1) = 1 \otimes_\sigma \mathbf{e}_2 \quad \psi_\infty(1 \otimes_\sigma \mathbf{e}_1) = \mathbf{e}_1 + \mathbf{e}_2$$
$$\varphi_\infty(\mathbf{e}_2) = p \otimes_\sigma \mathbf{e}_1 - 1 \otimes_\sigma \mathbf{e}_2 \quad \psi_\infty(1 \otimes_\sigma \mathbf{e}_2) = p\mathbf{e}_1$$

Outline

We have a map

$$\left\{ \begin{array}{c} \text{Kisin modules} \\ \text{relative to } \mathfrak{S} \to R \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \text{Kisin modules} \\ \text{relative to } \mathfrak{S} \to k[[t]] \end{array} \right\}$$

$$(M, \varphi, \psi) \mapsto (M, \varphi_\infty, \psi_\infty)$$

$$\varphi_\infty(x) = \epsilon_0(\varphi(x)))$$

$$\psi_\infty(y) = \epsilon_0(\psi(y)))$$

where $\epsilon_0 : \mathfrak{S} \to W$ is evaluation at $u = 0$.

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$$(M, \varphi, \psi) \mapsto (M, \varphi_\infty, \psi_\infty)$$

$$\varphi_\infty(x) = \epsilon_0(\varphi(x)))$$

$$\psi_\infty(x) = \epsilon_0(\psi(x)))$$

where $\epsilon_0 : \mathfrak{S} \to W$ is evaluation at $u = 0$.

Issues:

1. I doubt this map is onto.
2. I know this map is not one-to-one: $M = \mathfrak{S}_n \mathbf{e}, \varphi(\mathbf{e}) = u \otimes_\sigma \mathbf{e}$ and $M = \mathfrak{S}_n \mathbf{e}, \varphi(\mathbf{e}) = u^2 \otimes_\sigma \mathbf{e}$ both give $M = \mathfrak{S}_n \mathbf{e}, \varphi_\infty(\mathbf{e}) = 0, \psi_\infty(\mathbf{e}) = p\mathbf{e}$.
3. This seems unnatural when the whole tower does not lift.
4. Hard to determine the corresponding Hopf algebras, particularly over $k[[t]]$.

Example (Very Last Example)

Let $C_p = \langle \tau \rangle$. Let $M = \mathfrak{S}_1 \mathbf{e}, \varphi(\mathbf{e}) = u^{e-(p-1)} \otimes_\sigma \mathbf{e}, \psi(1 \otimes_\sigma \mathbf{e}) = u^{p-1}\mathbf{e}$.
Then $H_M = R[\frac{\tau-1}{\pi}] \subset KC_p$.
Let $M_1 = \mathfrak{S}_1 \mathbf{e}, \varphi_1(\mathbf{e}) = u^{p(e-(p-1))} \otimes_\sigma \mathbf{e}, \psi_1(1 \otimes_\sigma \mathbf{e}) = u^{p(p-1)}\mathbf{e}$. Then $H_{M_1} = R_1[\frac{\tau-1}{\pi_1^p}] = R_1[\frac{\tau-1}{\pi}] \subset K_1 C_p$.
Generally,

$$H_{M_m} = R_m[\frac{\tau-1}{\pi_m^{p^m}}] = R_m[\frac{\tau-1}{\pi}] \subset K_m C_p.$$

Let $m \to \infty$. Then

$$\varphi_\infty(\mathbf{e}) = 0 \otimes_\sigma \mathbf{e} = 0, \psi_\infty(1 \otimes_\sigma \mathbf{e}) = 0 = 0.$$

What's H_{M_∞}?

Thank you.