Perfectoid spaces: thoughts and conjectures

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1 Motivation

Let k be a local field with residue characteristic p and residue field $\mathbb{F}_q$, and let X be a smooth proper variety over k. Choose a prime ℓ distinct from the characteristic of k. Let $V = H^i_{et}(X \times_k \bar{k}, \bar{\mathbb{Q}}_\ell)$. Then the absolute Galois group G_k acts continuously on V, i.e. we have $\rho : G_k \to \mathrm{GL}(V)$. Let $I \subset G_k$ be the inertia subgroup, and let $t : I \to \mathbb{Z}_\ell$ be the canonical map determined by $\sigma(\pi^{1/\ell^n}) = \zeta^{t(\sigma)}\pi^{1/\ell^n}$. Grothendieck's ℓ-adic monodromy theorem says that there is a (uniquely determined) nilpotent $\bar{\mathbb{Q}}_\ell$-linear map $N : V \to V$ and an open subgroup $J \subset I$ such that for all $\sigma \in J$, we have

$$\rho(\sigma) = \exp(Nt(\sigma))$$

(Since N is nilpotent, the infinite series makes sense.) This lets us define a canonical filtration $V_\bullet$ such that $NV_i \subset NV_{i-2}$ and $N : \mathrm{gr}^i V \xrightarrow{\sim} \mathrm{gr}^{-i} V$ for all $i \in \mathbb{Z}$. We now have

Conjecture 1 (Deligne)**.** *Let $\phi \in G_k$ be a geometric Frobenius. Then the eigenvalues α of $\rho(\phi)$ are all algebraic over $\mathbb{Q}$ and have $|\alpha| = q^{(i+j)/2}$.*

If k has characteristic p and X/k is induced by a curve $C/\mathbb{F}_q$, then Deligne proved that the weight-monodromy conjecture holds for X.

For X a geometrically connected projective smooth toric variety over k, Scholze was able to prove the weight-monodromy conjecture using Deligne's theorem and general facts relating the etale sites of X and something called the "tilt" $X^\flat$ of X.

2 Background

Let's start by recalling some of the basic notions of infinite Galois theory that Kevin Keating introduced. If K/k is an infinite field extension, give $\mathrm{Gal}(K/k)$ the *Krull topology*, where a basis of neighborhoods of 1 is given by the subgroups $\mathrm{Gal}(K/k')$ where k'/k ranges over all finite Galois subextensions. One can check that this yields an isomorphism (at least, if K/k is Galois):

$$\mathrm{Gal}(K/k) = \varprojlim_{\substack{k \subset k' \subset K \\ [k':k] < \infty}} \mathrm{Gal}(K/k')$$

This shows that $\mathrm{Gal}(K/k)$ is a *profinite group*, that is, it is an inverse limit of finite groups. The infinite version of the fundamental theorem of Galois theory states that there is an order-preserving bijection between *closed* subgroups of $\mathrm{Gal}(K/k)$ and intermediate subfields of K/k. In particular, if $K = k^s$, the separable closure of k, it is a fact that $G_k = \mathrm{Gal}(k^s/k)$ can be recovered from (and is, in some sense, equivalent to) the category of finite separable extensions of k.

With that background, I'm going to begin with a beautiful theorem due to Fontaine and Wintenberger that Kevin just missed talking about. There is an isomorphism of profinite groups:

$$G_{\mathbb{Q}_p(p^{1/p^\infty})} \simeq G_{\mathbb{F}_p((t))} \tag{1}$$

The field $\mathbb{Q}_p\left(p^{1/p^\infty}\right)$ is just the union $\bigcup_{n\geqslant 1}\mathbb{Q}_p\left(p^{1/p^n}\right)$. The isomorphism (1) is constructed relatively explicitly as follows. First, one notes the following canonical isomorphisms:

$$G_{\mathbb{Q}_p\left(p^{1/p^\infty}\right)} \simeq G_{\widehat{\mathbb{Q}_p\left(p^{1/p^\infty}\right)}}$$
$$G_{\mathbb{F}_p((t))} \simeq G_{\mathbb{F}_p((t^{1/p^\infty}))}$$

The field $\mathbb{F}_p((t^{1/p^\infty}))$ is just the completion of the perfect closure of $\mathbb{F}_p((t))$. One can readily check that a typical element of $\mathbb{F}_p((t^{1/p^\infty}))$ looks like

$$x = \sum_{r\in\mathbb{Z}[\frac{1}{p}]} x_r t^r$$

where $x_r \in \mathbb{F}_p$ and for each $N > 0$, the set $\{r \in \mathbb{Z}[\frac{1}{p}]_{\geqslant N} : x_r \neq 0\}$ is finite. Similarly, an element of $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ looks like

$$x = \sum_{r\in\mathbb{Z}[\frac{1}{p}]} [x_r] p^r$$

where $[\cdot] : \mathbb{F}_p \to \mathbb{Z}_p$ is the Teichmller character and the same condition on the x_r holds.

Write $k = \widehat{\mathbb{Q}_p\left(p^{1/p^\infty}\right)}$, $k^\flat = \mathbb{F}_p((t^{1/p^\infty}))$. The way one shows that $G_k \simeq G_{k^\flat}$ is by *explicitely* constructing a correspondence between finite separable extensions of k and $k^\flat$. (*Note*: In [1], Scholze calls $k^\flat$ the *tilt* of k.) I will describe this correspondence in some generality, discuss Scholze's reasons for considering it, and then go on to make some conjectures.

Call a field k with a rank-1 valuation $v : k^\times \to \mathbb{Q}$ and residue characteristic $p > 0$ a *perfectoid field* if

- k is complete
- the value group $v(k^\times)$ is non-discrete
- the Frobenius is surjective on $\mathcal{O}_k/p$.

In particular, note that a perfectoid field of characteristic p is a perfect field. Our fields $k = \widehat{\mathbb{Q}_p\left(p^{1/p^\infty}\right)}$ and $k^\flat = \mathbb{F}_p((t^{1/p^\infty}))$ are both perfectoid fields. Similarly, the fields $\mathbb{C}_p$ and $\widehat{\overline{\mathbb{F}_p((t))}}$ are perfectoid. I currently do not have a good feel for which infinite extensions $k/\mathbb{Q}_p$ are perfectoid.

3 The tilting functor

Let k be a perfectoid field. Scholze defines a new perfectoid field $k^\flat$ of characteristic p. If k already has characteristic p, then $k = k^\flat$. In general, $k^\flat$ has the same value group as k, the same residue field, and there is a canonical isomorphism $G_k \simeq G_{k^\flat}$.

Start by considering the ring

$$\mathcal{R} = \varprojlim_{x\mapsto x^p} \mathcal{O}_k/(p) = \left\{(\bar{x}_i)_{i\geqslant 1} : \bar{x}_{i+1}^p = \bar{x}_i\right\}$$

The ring $\mathcal{R}$ is clearly of characteristic p (note $p \cdot (\bar{x}_i) = (p\bar{x}_i) = 0$), and is in fact perfect. For, one has

$$(\bar{x}_1, \bar{x}_2, \dots)^{1/p} = (\bar{x}_2, \bar{x}_3, \dots)$$

One can check that $\mathcal{R}$ is in fact an integral domain, and sets $k^\flat$ to be its field of fractions. There is a canonical multiplicative (but not additive) isomorphism $\sharp : k^\flat \to k$, defined as follows on $\mathcal{R}$:

$$x^\sharp = (\bar{x}_1, \bar{x}_2, \dots)^\sharp = \lim_{i\to\infty} x_i^{p^i}$$

where the x_i are arbitrary lifts of $\bar{x}_i$ to $\mathcal{O}_k$. The correspondence

$$\{\text{finite extensions of } k\} \to \left\{\text{finite extensions of } k^\flat\right\}$$

is given by $K/k \mapsto K^\flat/k^\flat$. Scholze claims (and I haven't fully worked this out) that the inverse is given by

$$K/k^\flat \mapsto K^\sharp = W\left(\mathcal{O}_K\right) \otimes_{W\left(\mathcal{O}_{k^\flat}\right)} k$$

4 An example

Let's work out an example. As before, let $k = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$. One has

$$\mathcal{O}_k/(p) = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}/(p) = \mathbb{F}_p[t^{1/p^\infty}]/(t)$$

One can construct an isomorphism $\mathbb{F}_p[\![t^{1/p^\infty}]\!] \to \varprojlim_{x \mapsto x^p} \mathbb{F}_p[t^{1/p^\infty}]/(t)$ by sending t^r to the sequence $(t^r, t^{r/p}, t^{r/p^2}, \dots)$. (Some of the first few terms may be zero, but that's okay.) I've tried to get a feel for what the inverse part of the correspondence does. Note that

$$W(\mathcal{O}_{k^\flat}) = W(\mathbb{F}_p[\![t^{1/p^\infty}]\!]) = \mathbb{Z}_p[\![t^{1/p^\infty}]\!]$$

One can check that the canonical map $W(\mathcal{O}_{k^\flat}) \to k$ is just the map $t \mapsto p$. It then looks like we can start with an extension K_0 of $\mathbb{F}_p(\!(t)\!)$. Suppose $\mathcal{O}_{K_0} = \mathbb{F}_p[\![t]\!][x]$, where f is the minimal polynomial of x. Let $K = \mathbb{F}_p(\!(t^{1/p^\infty})\!)[x]$; then $\mathcal{O}_K = \mathcal{O}_{k^\flat}[X]/(f)$. If $\tilde{f}$ is any lift of f to $\mathbb{Z}[\![X]\!]$, then (I think) $W(\mathcal{O}_K) = \mathbb{Z}_p[\![t^{1/p^\infty}]\!][X]/(\tilde{f})$, so

$$K^\sharp = \mathbb{Z}_p[\![t^{1/p^\infty}]\!][X]/(\tilde{f}) \otimes_{\mathbb{Z}_p[\![t^{1/p^\infty}]\!]} \widehat{\mathbb{Q}_p(p^{1/p^\infty})} = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}[X]/(\tilde{f}|_{t=p})$$

5 Conjectures

As before, let $k = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $k^\flat = \mathbb{F}_p(\!(t^{1/p^\infty})\!)$. The isomorphism $G_k \simeq G_{k^\flat}$ essentially says that

Categorical statements about field extensions are true over k iff they are true over $k^\flat$.

My first conjecture (more accurately, heuristic towards a conjecture) is that "categorical statements true over $k^\flat$ become true over k_n for $n \gg 0$." Here $k_n = \mathbb{Q}_p(p^{1/p^n})$. How to make this precise? Say the *elementary first-order theory of categories* Σ involves all statements built up as follows (where C is our base category):

- If $\varphi, \psi \in \Sigma$, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, and $\neg\varphi$ are in Σ.
- If $\varphi \in \Sigma$, then $\forall x \in C : \varphi$, $\exists x \in C : \varphi$, $\forall f : x \to y : \varphi$, and $\exists f : x \to y : \varphi$ are in Σ.
- Statements of the form $f \circ g = h$ are in Σ.
- Variables for morphisms and objects are in Σ.

For example, the statement "$\mathrm{Gal}(K/k) = \mathbb{Z}/3$" is in Σ (over the category of etale k-algebras), as it can be written as

$$\exists f_1, f_2, f_3 : \forall f : K \to K : (f = f_1 \vee f = f_2 \vee f = f_3) \wedge (f_1 \circ f_1 = f_1, f_1 \circ f_2 = f_2, \dots)$$

here ... represents the identities required to force $\mathrm{Gal}(K/k) = \mathbb{Z}/3$. Statements not in Σ include statements that have quantification like

$$\forall x_1, \dots, x_n : \varphi$$

where n is variable, not a fixed integer. So for instance, the statement "$\mathrm{Gal}(K/k)$ is cyclic" is *not* in Σ, although the statement "$\mathrm{Gal}(K/k)$ is abelian" is in Σ.

Conjecture 2. *If $\varphi \in \Sigma$ issuch that φ is true for the category of etale $k^\flat$-algebras, then there exists n_0 such that for all $n \geqslant n_0$, φ is true for the category of etale k_n-algebras.*

The next conjecture is much more vague. Let φ be a statement in the "first-order language of rank-one valuations," where such a language is not at the moment well-defined. If we denote by V such a language, the sort of statements I think should be in V are

- $v(f) \geq r$, where f is an expression and $r \in \mathbb{Q}$ is a fixed rational number,
- $v(f) \geqslant v(g)$, where f, g are expressions,
- any polynomial expression (with a fixed number of variables).

So for example, something like

$$v((x+y)^5 - x^5 - y^5) \geqslant 1239472$$

should be in V, while something like

$$\forall r \in \mathbb{Q} : v((x+y)^7 - x^7 - y^7) > r$$

should not. The following conjecture is not precisely-stated at the moment.

Conjecture 3. *If $\varphi \in V$ holds over all complete rank-one valued fields of characteristic p, then let $x_1, \ldots, x_n$ be the variables quantified over in φ. There exists $f = f(v(p), v(x_1), \ldots, v(x_n))$ such that for all characteristic zero rank-one valued fields of residue characteristic p and $v(p) > f$, φ holds.*

One other natural problem is to try and relate Scholze's theory of perfectoid fields with Fontaine and Wintenberger's theory of arithmetically profinite fields. For example, when is the completion of an algebraic extension $k/\mathbb{Q}_p$ perfectoid? Is it whenever $k/\mathbb{Q}_p$ is arithmetically profinite? Moreover, if $k/\mathbb{Q}_p$ is arithmetically profinite and perfectoid, is $k^\flat$ the completion of the perfect closure of the field of norms $X(k/\mathbb{Q}_p)$? This is certainly the case if $k = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$.

References

[1] Scholze, Peter. *Perfectoid spaces*, `arXiv:1111.4914v1`.