Quasitriangular structures on (dual) monoid rings

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Background/Literature

- R.G. Underwood, *Topics in Modern Algebra - A View Towards Hopf Algebras* (forthcoming)
  which gives an exposition of results from
- W.D. Nicholls, R.G. Underwood, *Algebraic Myhill-Nerode Theorems*, Theoretical Computer Science **412** (2011) 448–457.
- R.G. Underwood, *Quasitriangular structure on Myhill-Nerode Bialgebras*, Axioms **1**, 2012, 155-172.

Monoids

A **monoid** is a set M with a multiplication $*$ such that

- $x * (y * z) = (x * y) * z$ for all x, y, $z \in M$;
- there exists $1 \in M$ with $x * 1 = x = 1 * x$ for all $x \in M$.

(Like a group, but no inverses.)

Note we do not necessarily have cancellation:

$$x * y = x * z \not\Rightarrow y = z.$$

Monoids: Examples

- any group;
- $\mathbb{Z}$ under multiplication;
- finite words in some alphabet;
- $T_n = \{0, 1, \ldots, n\}$ with

$$i * j = \begin{cases} i + j & \text{if } i + j \leq n; \\ n & \text{if } i + j \geq n. \end{cases}.$$

So $x * y = 0 \Rightarrow x = y = 0$, and $x * n = n$ for all x.

Table for T_4:

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	4
2	2	3	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

Bialgebras from Monoids

Just as there are two standard ways to get a Hopf algebra from a finite group, we can get bialgebras (no antipode!) from a finite monoid M.

Let k be an integral domain (e.g. a field). We have

- The **monoid ring**

$$k[M] = \left\{ \sum_{x \in M} c_x x \ : \ c_x \in M \right\},$$

with

$$\Delta \left(\sum_{x \in M} c_x x \right) = \sum_{x \in M} c_x x \otimes x$$

(always cocommutative);

- the **dual monoid ring** $\mathrm{Map}(M, k)$, with k-basis $\{e_x \ : \ x \in M\}$, where

$$e_x(y) = \delta_{x,y}; \quad e_x e_y = \delta_{x,y} e_x; \quad \Delta(e_x) = \sum_{y*z=x} e_y \otimes e_z$$

(always commutative).

Quasitriangular Structures

Let B be a k-bialgebra. A **quasitriangular structure** (QTS) on B is an element $R = \sum_i a_i \otimes b_i \in (B \otimes B)^\times$ such that

(1) (almost cocommutativity)

$$\tau(\Delta(b))R = R\Delta(b) \text{ for all } b,$$

where $\tau(x \otimes y) = y \otimes x$;

(2a)

$$(\Delta \otimes \mathrm{id})R = R^{13}R^{23};$$

(2b)

$$(\mathrm{id} \otimes \Delta)R = R^{13}R^{12};$$

where $R^{13} = \sum_i a_i \otimes 1 \otimes b_i$, etc.

These arise in constructing representations of braid groups, solutions of the Yang-Baxter equation in statistical mechanics, etc.

Quasitriangular Structures

Remarks:

(a) If B is cocommutative, $R = 1 \otimes 1$ is a QTS, and the QTS on B form a subgroup of $(R \otimes R)^{\times}$.

(b) If B is commutative but **not** cocommutative, the almost cocommutativity condition (1) is **never** satisfied, so there are no QTS.

To make things more interesting, we make the (non-standard) definition:

A **weak QTS** is $R \in (B \otimes B)^{\times}$ satisfying (2a) and (2b) but not necessarily (1).

Finite Automata

We now venture into theoretical computer science.

A finite automaton reads in words from some alphabet, changing state according to the symbol just read and its current state.

The word is “accepted” if the automaton is in one of the “accepting” states.

Finite Automata

The words accepted by the automaton form a **language** L, i.e. a subset of the monoid of words in the alphabet, e.g. all words not containing 2 consecutive b's.

Given a language L, is there a finite automaton accepting it?

Define an equivalence relation on words: $W \sim_L W'$ precisely when

$$Wx \in L \Leftrightarrow W'x \in L \text{ for all symbols } x \in \Sigma.$$

Theorem (Myhill-Nerode):
There is a finite automaton accepting L if and only if $\sim_L$ has only finitely many equivalence classes.

Bialgebra approach to proof (Nicholls-Underwood)

Let S be the (infinite) monoid on Σ, let $H = k[S]$ be its monoid ring, and $p \in H^* = \mathrm{Hom}_k(H, k)$, where, for $w \in S$,

$$p(w) = \begin{cases} 1 & \text{if } w \in L; \\ 0 & \text{if } w \notin L. \end{cases}$$

For $w \in S$ and $q \in H^*$, define $q \leftharpoondown w \in H^*$ by

$$(q \leftharpoondown w)(y) = q(yw) \text{ for } y \in S.$$

Then $\sim_L$ has finitely many equivalence classes if and only if $Q := \{p \leftharpoondown w : \ w \in S\}$ is finite.

Bialgebra approach to proof (Nicholls-Underwood)

So if $\sim_L$ has finitely many equivalence classes, we can build a finite automaton with Q as its set of states.

Then S acts on Q via a finite quotient monoid whose elements are functions $Q \longrightarrow Q$.

Thus any finite automaton gives rise to a finite monoid.

We call these **Myhill-Nerode monoids** and their monoid rings **Myhill-Nerode bialgebras**.

Our Previous Example;:

For the finite automaton detecting two consecutive b's, the monoid is

$$M = \{1, b, b^2, a, ab, ba\}$$

with multiplication table

	1	b	b^2	a	ab	ba
1	1	b	b^2	a	ab	ba
b	b	b^2	b^2	ba	b	b^2
b^2	b^2	b^2	b^2	b^2	b^2	b^2
a	a	ab	b^2	a	ab	a
ab	ab	b^2	b^2	a	ab	b^2
ba	ba	b	b^2	ba	b	ba

A Very Simple Example and a Question:

Fix $n \geq 1$. Let

$$\Sigma = \{a\}, \qquad L = \{a^{n-1}\}.$$

This gives an automaton with states 0, 1, ..., n, with $n-1$ as the only accepting state. The corresponding monoid is T_n of size $n+1$.

The monoid ring $k[T_n]$ and its dual $\mathrm{Map}(T_n, k)$ are commutative and cocommutative.
They are isomorphic as bialgebras for $n = 1$ but not for $n \geq 2$.

Question (Underwood):
What quasitriangular structures are possible on these bialgebras?

Theorem (Underwood):
For $n = 1$, the only QTS is $R = 1 \otimes 1$.

The answer

Theorem (NB):
For all $n \geq 1$,

(i) the only QTS on $k[T_n]$ is $1 \otimes 1$;

(ii) the only QTS on $\mathrm{Map}(T_n, k)$ is $1 \otimes 1$.

QTS on Dual Monoid Rings

Let $B = \mathrm{Map}(M, k)$ for a finite monoid M. Consider

$$R = \sum_{x,y \in M} \alpha(x, y) e_x \otimes e_y \in (B \otimes B)^\times.$$

Thus $\alpha(x, y) \in k^\times$ for all x, y. Then

$$\begin{aligned} & (\Delta \otimes \mathrm{id})R = R^{13} R^{23} \\ \Leftrightarrow \quad & \alpha(xy, z) = \alpha(x, z)\alpha(y, z) \quad \forall x, y, z \\ \Leftrightarrow \quad & \quad \forall z, \text{ the map } x \mapsto \alpha(x, z) \\ & \qquad \text{is a monoid homomorphism } M \longrightarrow k^\times. \end{aligned}$$

(Then its image is a finite *subgroup* of $k^\times$.)

Similarly for $(\mathrm{id} \otimes \Delta)R = R^{13} R^{12}$.

QTS on Dual Monoid Rings

So we have proved:

Lemma:
The weak QTS on $\mathrm{Map}(M, k)$ correspond to bicharacters
$\alpha\colon M \times M \longrightarrow k^\times$,
i.e. functions such that $\alpha(\cdot, x)$ and $\alpha(x, \cdot)$ are monoid homomorphisms $\forall x$.

Example:
For a homomorphism $\theta\colon T_n \longrightarrow k^\times$,

$$\theta(i)\theta(n) = \theta(i * n) = \theta(n) \Rightarrow \theta(i) = 1.$$

Hence the only QTS on the bialgebra $\mathrm{Map}(T_n, k)$ is $R = 1 \otimes 1$.

QTS on Dual Monoid Rings

For our non-commutative monoid $M = \{1, b, b^2, a, ab, ba, b^2\}$,

	1	b	b^2	a	ab	ba
1	1	b	b^2	a	ab	ba
b	b	b^2	b^2	ba	b	b^2
b^2	b^2	b^2	b^2	b^2	b^2	b^2
a	a	ab	b^2	a	ab	a
ab	ab	b^2	b^2	a	ab	b^2
ba	ba	b	b^2	ba	b	ba

the bialgebra $\mathrm{Map}(M, k)$ admits no QTS since it is commutative but not cocommutative.

What about *weak* QTS? For a homomorphism $\theta \colon M \longrightarrow k^\times$,

$$\theta(b^2)\theta(x) = \theta(b^2 * x) = \theta(b^2) \quad \forall x,$$

so $\theta(x) = 1 \quad \forall x$.

Hence the only weak QTS, $R = 1 \otimes 1$.

QTS on Monoid Rings

QTS on $k[M]$ are not so transparent. Let

$$R = \sum_{x,y \in M} \beta(x,y) x \otimes y.$$

Then

$$\begin{aligned} & (\Delta \otimes \mathrm{id})R = R^{13}R^{23} \\ \Leftrightarrow \quad & \sum_{u * v = z} \beta(x,u)\beta(y,v) = \beta(x,z)\delta_{x,y} \quad \forall x, y, z. \end{aligned}$$

Similarly for $(\mathrm{id} \otimes \Delta)R = R^{13}R^{12}$.

QTS on Monoid Rings

Apply this to T_n.
Taking $x = y = z = 0$ in

$$\sum_{u*v=z} \beta(x,u)\beta(y,v) = \beta(x,z)\delta_{x,y}$$

gives $\beta(0,0)^2 = \beta(0,0)$, so $\beta(0,0) = 0$ or 1.

But if $\beta(0,0) = 0$ then $R \notin (B \otimes B)^\times$. So $\beta(0,0) = 1$.

Taking $x = j \neq 0$, $y = z = 0$ gives

$$\beta(j,0)\beta(0,0) = 0 \Rightarrow \beta(j,0) = 0 \quad \forall j.$$

Similarly $\beta(0,j) = 0 \quad \forall j$.

QTS on Monoid Rings

If $1 \leq i \leq n$ and $1 \leq j \leq n-1$ with $\beta(i,h) = 0$ for $1 \leq h < j$, then

$$\beta(i,j) = \sum_{h=0}^{j} \beta(i,h)\beta(i,j-h) = 0.$$

Similarly for $\beta(j,i)$.
So, inductively, $\beta(i,j) = 0$ unless $(i,j) = (0,0)$ or (n,n).

Finally, taking $x = 0$, $y = n$, $z = n$:

$$0 = \sum_{u+v \geq n} \beta(0,u)\beta(n,v) = \beta(0,0)\beta(n,n) = \beta(n,n).$$

Hence the unique QTS on $\mathrm{Map}(T_n, k)$ is $R = 1 \otimes 1$.

Remaining Questions

- For $M = \{1, b, b^2, a, ab, ba\}$, is $1 \otimes 1$ the only QTS on $k[M]$? Is it the only weak QTS?
- What happens for (finite, nonabelian) groups, e.g. S_3?

  $\mathrm{Map}(S_3, k)$ has no QTS and two weak QTS (provided k does not have characteristic 2), coming from the maximal abelian quotient $S_3/A_3 = C_2$.

  Are there any QTS (or weak QTS) on $k[S_3]$ apart from $1 \otimes 1$?