Hopf orders in elementary abelian group rings

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May 22, 2014

Any introduction to Hopf algebras will point out that the group ring $K[G]$ is a Hopf algebra with comultiplication,

$$\Delta : K[G] \to K[G] \otimes K[G],$$

the K-algebra map determined by

$$\Delta(\sigma) = \sigma \otimes \sigma \text{ for all } \sigma \in G.$$

We are interested in a situation where K is a local field. So:

- p, a prime number
- K, a local field complete w.r.t a normalized valuation v_K with residue field κ of characteristic p. $v_K(\pi) = 1$.
  for example,
  - Char. p: $K = \kappa((t))$ (*e.g.* $\pi = t$)
  - Char. 0: K, a finite extension of the p-adic numbers $\mathbb{Q}_p$
- G, a p-group. $|G| = p^n$ for some $n \geq 1$

- valuation ring/ring of integers: $\mathfrak{O}_K = \{x \in K : v_K(x) \geq 0\}$

What *Hopf order* lie in $K[G]$?

Find all $\mathfrak{O}_K$-modules H within $K[G]$ that have full rank, namely $|G|$, *but* are also Hopf algebras over $\mathfrak{O}_K$.
i.e. *closed under comultiplication*, $\Delta : H \to H \otimes H$.

(Tate/Oort 1970) gave a complete classification of these Hopf orders when $K/\mathbb{Q}_p$ is finite and $|G| = p$.

...seemingly setting the direction of future research. *So that only recently have fields with char. p been considered.*

Before we proceed, a Fact:

Hopf orders in $K[G]$ contain the group ring $\mathfrak{O}_K[G]$.

(Larson 1976) develops p-adic, order-bounded *group valuations*. These are functions $v : G \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ that satisfy 5 properties, making H, the $\mathfrak{O}_K$-span of

$$\left\{ \frac{\sigma - 1}{\pi^{v(\sigma)}} : \sigma \in G \right\},$$

into a Hopf order. There are two things to check:

Closure under comultiplication comes "for free" since

$$\Delta \left(\frac{\sigma - 1}{\pi^{v(\sigma)}} \right) = \frac{\sigma \otimes \sigma - 1 \otimes 1}{\pi^{v(\sigma)}} = \frac{\sigma - 1}{\pi^{v(\sigma)}} \otimes \sigma + 1 \otimes \frac{\sigma - 1}{\pi^{v(\sigma)}},$$

and $1, \sigma \in \mathfrak{O}_K[G] \subseteq H$.

So the 5 conditions are required for closure under multiplication.

Restrict to elementary abelian groups...

Continuing with $K/\mathbb{Q}_p$ finite...

When G is elementary abelian of order p^n, the group valuation conditions are largely vacuous, and Larson orders look like

$$H = \mathfrak{O}_K \left[\frac{\sigma_1 - 1}{\pi^{M_1}}, \dots, \frac{\sigma_n - 1}{\pi^{M_n}} \right]$$

where $G = \langle \sigma_1, \dots, \sigma_n \rangle$ with

$$0 \leq M_i \leq \frac{v_K(p)}{p-1}.$$

- Tate/Oort: When $n = 1$, every Hopf order is a Larson order.

When $n = 2$, the Hopf orders were classified completely by (Greither, 1992) and (Byott, 1993). They looked like

$$H_2 = \mathfrak{O}_K \left[\frac{\sigma_1 - 1}{\pi^{M_1}}, \frac{\sigma_2 u_2 - 1}{\pi^{M_2}} \right]$$

for certain $u_2 \in H_1 = \mathfrak{O}_K[(\sigma_1 - 1)/\pi^{M_1}]$ satisfying two closure conditions:

$$u_2^p \in \pi^{pM_1} H_1 \text{ and } \Delta(u_2) \in \pi^{M_1} H_1 \otimes H_1.$$

(Greither/Childs, 1998) and then (Childs/Smith III, 2005) inductively exhibited families of Hopf orders in $K[C_p^n]$ for $n \geq 3$

$$H_n = \mathfrak{O}_K \left[\frac{\sigma_1 - 1}{\pi^{M_1}}, \frac{\sigma_2 u_2 - 1}{\pi^{M_2}}, \dots, \frac{\sigma_n u_n - 1}{\pi^{M_n}} \right]$$

with $u_i^p \in \pi^{pM_{i-1}} H_{i-1}$ and $\Delta(u_i) \in \pi^{M_{i-1}} H_{i-1} \otimes H_{i-1}$.

But conclude that their classification is incomplete, even for $n = 3$.

Missing are "realizable Hopf orders"...

So far, K is still a finite extension of $\mathbb{Q}_p$, and it is important to note that all the descriptions of u_i bound $v_K(p)$ from below.

In characteristic p, $v_K(p) = \infty$, which suggests that the results ought to hold in characteristic p as well. Except...

The descriptions of these u_i involve the p distinct pth roots of unity, as they are generalizations of so-called Greither orders.

We need another "language"

Galois extension of local fields

Given

$$\begin{array}{ccc} L & & \mathfrak{O}_L \\ | & G = \mathrm{Gal}(L/K) & | \\ K & & \mathfrak{O}_K \end{array}$$

there is the associated order

$$\mathcal{A}_{L/K} = \{\alpha \in K[G] : \alpha\mathfrak{O}_L \subseteq \mathfrak{O}_L\}.$$

Assuming G abelian and some mild conditions, (Bondako, 2000) proves that $\mathfrak{O}_L$ is free over $\mathcal{A}_{L/K}$ iff $\mathcal{A}_{L/K}$ is a Hopf order.

Recently, Byott and I developed a tool, *scaffold*, with which we are able to exert some fairly explicit control over the way that $\mathcal{A}_{L/K}$ acts on $\mathfrak{O}_L$, as well as provide an explicit basis for $\mathcal{A}_{L/K}$.

Define $(1+X)^{[Y]} = \sum_{i=0}^{p-1} \binom{Y}{i} X^i \in \mathbb{Z}_{(p)}[X, Y]$.

Realizable Hopf orders in $K[C_p^3]$ from scaffolds

From (Byott/Elder, preprint), using (Byott/Childs/Elder, preprint)

$$\mathfrak{O}_K\left[\frac{\sigma_1-1}{\pi^{M_1}},\frac{\sigma_2\sigma_1^{[\mu_{2,1}]}-1}{\pi^{M_2}},\frac{\sigma_3\sigma_1^{[\mu_{3,1}]}\left(\sigma_2\sigma_1^{[\mu_{2,1}]}\right)^{[\mu_{3,2}]}-1}{\pi^{M_3}}\right]$$

is a Hopf order in $K[C_p^3]$ for all $M_i \geq 0$ and $\mu_{j,i} \in K$ satisfying

$$\frac{v_K(p)}{p-1} > M_1+M_2+M_3, \quad p^2M_1 \geq pM_2 \geq M_3 > 0, \text{ and}$$

$$v_K(\mu_{j,i}) = M_j - \frac{M_i}{p^{j-i}},$$

and some additional conditions, including the existence of $\omega_2, \omega_3 \in K$ with $v_K(\omega_3) \leq v_K(\omega_2) \leq 0$ with $\omega_2^p \not\equiv \omega_2 \bmod \mathfrak{P}_K$ such that

$$\mu_{1,2} = \omega_2, \quad \mu_{2,3} = \frac{\omega_3^p-\omega_3}{\omega_2^p-\omega_2}, \quad \mu_{1,3} = \frac{\omega_2\omega_3^p-\omega_3\omega_2^p}{\omega_2^p-\omega_2}.$$

Project: Characteristic-free description of Hopf orders

Let $G = \langle \sigma_1, \ldots, \sigma_n \rangle \cong C_p^n$. Given $M_i \geq 0$, and $\mu_{j,i} \in K$ such that $0 \geq v_K(\mu_{j,i}) \geq -M_i$, we are interested in further conditions so that

$$H_n = \mathfrak{O}_K \left[\frac{\Theta_1 - 1}{\pi^{M_1}}, \ldots, \frac{\Theta_n - 1}{\pi^{M_n}} \right]$$

where $\Theta_1 = \sigma_1$ and recursively

$$\Theta_i = \sigma_i \overbrace{\Theta_1^{[\mu_{i,1}]} \cdots \Theta_{i-1}^{[\mu_{i,i-1}]}}^{u_i}$$

is a Hopf order in $K[G]$.

Conditions, defined inductively, such that $u_i^p \in \pi^{pM_{i-1}} H_{i-1}$ and $\Delta(u_i) \in \pi^{M_{i-1}} H_{i-1} \otimes H_{i-1}$

Results

Let $\wp(x) = x^p - x$, $\mathfrak{M}_{j,j-1} = \wp(\mu_{j,j-1})$, and recursive define

$$\mathfrak{M}_{j,i} = \wp(\mu_{j,i}) + \sum_{s=i+1}^{j-1} \mu_{j,s}\mathfrak{M}_{s,i} \in K.$$

Theorem:

Char. p

- In characteristic p, closure under multiplication is "for free"
- If $v_K(\mathfrak{M}_{j,i}) \geq M_j - pM_i$, and $p \geq n$, then we have closure under comultiplication, **and** H_n is a Hopf order in $K[C_p^n]$.

Char. 0

- Closure under multiplication follows from

$$\frac{v_K(p)}{p-1} \geq M_1 + M_2 + \cdots + M_n$$

- Conjecture: The conditions above are sufficient for closure under comultiplication as well, and thus H_n is a Hopf order.

For $n = 3$, the Hopf orders from scaffolds conditions become:
$v_K(\mathfrak{M}_{2,1}) = M_2 - pM_1$, $v_K(\mathfrak{M}_{3,2}) = M_3 - pM_2$, and $\mathfrak{M}_{3,1} = 0$.

Comments

- Here again char. 0 approximates char. p (*i.e.* just assume $v_K(p)$ is large enough).
- Every Hopf order H determines a group valuation v, that in turn determines a Larson order contained within H. I would like to be able to start with a Hopf order, determine the group valuation, determine the $\mu'_{j,i}s \in K$ (I need a language for what these $\mu_{j,i}$ "represent"). *So that* using the invariants (group valuation and $\mu_{j,i}$'s), we can create the largest "characteristic independent order in H. Then, at least in characteristic p, there should be a discriminant argument that equates the two objects, giving the classification in characteristic p.
- In char. 0 we expect the picture to be more complicated. For $n = 2$ and $G \cong C_{p^2}$, (Underwood, 1996) showed that Hopf orders are either Greither order, or the duals of Greither orders or both. Every Greither order can be written in the characteristic independent language that we have just seen. And assuming the presence of pth roots on unity, the reverse holds.

Wondering out loud

- Could it be the case that Hopf orders, in char. 0, can either be described by this characteristic independent description with

$$M_1 + M_2 + \cdots + M_n < \frac{v_K(p)}{p-1},$$

or, if not, are the dual of some Hopf order that can?

Seems unlikely given that the Larson restriction is to the hypercube defined by $0 \leq M_i \leq \frac{v_K(p)}{p-1}$, and for $n > 2$, slicing through those vertices that are adjacent to the origin cuts off a corner that is less than half of the whole.