Parametrizations of Local Field Extensions

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Reference

M. Fried and A. Mézard
"Configuration spaces for wildly ramified covers"
Arithmetic Fundamental Groups and Noncommutative Algebra (Berkeley, CA, 1999)
Proceedings of Symposia in Pure Mathematics **70**
American Mathematical Society, Providence, RI, 2002
pp. 353–376

Notation

$k =$ algebraically closed field of characteristic $p > 0$

$F = k((x)) =$ local field of characteristic p

$\mathcal{O}_F = k[[x]] =$ ring of integers of F

$\overline{F} =$ algebraic closure of F

$E/F =$ finite subextension of $\overline{F}/F$

All "extensions" are finite extensions of F.

Say extensions E, E' are isomorphic if they are isomorphic as extensions of F.

Series Associated to Extensions

Let E/F be a finite extension, and let π_E be a uniformizer for E. Then there is a unique series

$$g(T) = \sum_{i=1}^{\infty} a_i T^i \in k[[T]]$$

such that $x = g(\pi_E)$.

If π_E' is another uniformizer for E then there is

$$\gamma(T) = b_0 T + b_1 T^2 + \cdots \in k[[T]]$$

with $b_0 \neq 0$ such that $\pi_E = \gamma(\pi_E')$. Hence $x = g(\gamma(\pi_E'))$.

It follows that the series $g(\gamma(T))$ is also associated to the extension E/F.

Extensions Associated to Series

Let $n \geq 1$ and let $g(T) = \sum_{i=1}^{\infty} a_i T^i \in T^n \cdot k[[T]]^\times$.

By WPT, $g(T) - x = u(T)h(T)$, with

- $u(T) \in \mathcal{O}_F[[T]]^\times$
- $h(T) \in \mathcal{O}_F[T]$ Eisenstein of degree n.

Let $y \in \overline{F}$ be a root of $h(T)$. Then $F(y) \cong k((y))$ is a (totally ramified) extension of F of degree n.

The F-isomorphism class of $F(y)$ does not depend on the choice of root y.

Let $\mathrm{Ext}(g)$ denote the F-isomorphism class of the extension $F(y)$ of F associated to $g(T)$.

Action of $\mathcal{A}(k)$ on $T \cdot k[[T]]$

The set $\mathcal{A}(k) = T \cdot k[[T]]^\times$ is a group under the operation of substitution.

$\mathcal{A}(k)$ acts on the set $T \cdot k[[T]]$ on the right by substitution. Let $g(T) \in T \cdot k[[T]]$ and $\gamma(T) \in \mathcal{A}(k)$. Then

$$g \cdot \gamma = g(\gamma(T)).$$

Set $\tilde{g}(T) = g(\gamma(T))$, and let $\tilde{g}(T) - x = \tilde{u}(T)\tilde{h}(T)$, with $\tilde{u}(T) \in F[[T]]^\times$ and $\tilde{h}(T)$ Eisenstein.

Let $\tilde{y} \in \overline{F}$ be a root of $\tilde{h}(T)$, and set $\tilde{E} = F(\tilde{y})$.

Then $\gamma(\tilde{y})$ is a root of $g(T) - x$, so $\tilde{E}$ is F-isomorphic to E.

Hence $\mathrm{Ext}(g) \cong \mathrm{Ext}(g \cdot \gamma)$, so $\mathcal{A}(k)$-orbits of $T \cdot k[[T]] \smallsetminus \{0\}$ correspond to isomorphism classes of finite extensions of F.

Ramification Data

For $i, j \in \mathbb{N}$ define

$$i \preceq j \Leftrightarrow i \leq j \text{ and } v_p(i) \leq v_p(j).$$

Given $g(T)$, let $A_g = \{i \in \mathbb{N} : a_i \neq 0\}$. The ramification data of $g(T)$ is the set $\mathrm{Ram}(g)$ of minimal elements of $(A_g, \preceq)$.

$\mathrm{Ram}(g)$ is finite, and forms an antichain in $(\mathbb{N}, \preceq)$.

Elements of $\mathrm{Ram}(g)$ correspond to nonzero terms $a_i T^i$ of $g(T)$ such that i and $v_p(i)$ are both small.

Say the finite nonempty set $D \subset \mathbb{N}$ is "valid ramification data" if $(D, \preceq)$ is an antichain.

Action of $\mathcal{A}(k)$ on $S(D)$

Let $D \subset \mathbb{N}$ be valid ramification data.

Set $S(D) = \{g(T) \in T \cdot k[[T]] : \mathrm{Ram}(g) = D\}$.

Then $S(D) \neq \{\,\}$.

For $\gamma \in \mathcal{A}(k)$ we have $\mathrm{Ram}(g \cdot \gamma) = \mathrm{Ram}(g)$.

It follows that

- $\mathrm{Ram}(g)$ depends only on the extension $\mathrm{Ext}(g)$.
- The group $\mathcal{A}(k)$ acts on $S(D)$.

We want to construct a small subset of $S(D)$ which has representatives from every orbit of this action.

Changing $g(T)$

Let $\delta_r(T) = T + zT^{r+1}$, with $z \in k$ to be determined.

Then $\delta_r \in \mathcal{A}(k)$.

What is the smallest degree term in $g(\delta_r(T)) - g(T)$?

If $i = up^j$ with $p \nmid u$ then

$$\delta_r(T)^i = T^i + uz^{p^j} T^{i+rp^j} + \cdots .$$

Hence small-degree terms in $g(\delta_r(T)) - g(T)$ come from nonzero terms $a_i T^i$ in $g(T)$ with i and $j = v_p(i)$ small.

It follows that the crucial terms are those that correspond to elements of $D = \mathrm{Ram}(g)$.

Changing $g(T)$, continued

Define $\Lambda_D : \mathbb{R} \to \mathbb{R}$ by

$$\Lambda_D(t) = \min\{d + p^{v_p(d)}t : d \in D\}.$$

Let $c = \Lambda_D(r)$. Then the terms in $g(\delta_r(T)) - g(T)$ all have degree $\geq c$.

We want z such that the coefficient of T^c in $g(\delta_r(T))$ is 0.

Let $q^- = \Lambda'_D(r - \epsilon)$ and $q^+ = \Lambda'_D(r + \epsilon)$.

Then

$$g(\delta_r(T)) - g(T) = h(z^{q^+})T^c + \cdots ,$$

with h a separable additive polynomial of degree q^-/q^+.

Hence there are q^-/q^+ values $z \in k$ which make the coefficient of T^c in $g(\delta_r(T))$ equal to 0.

Relation with the Usual Ramification Data

Suppose $\mathrm{Ram}(g) = D$, and set $E = \mathrm{Ext}(g)$.

Then D can be computed from $n = [E : F]$ and the indices of inseparability $i_0, i_1, \ldots, i_\nu$ of E/F:

$$D = \{i_0 + n, i_1 + n, \ldots, i_\nu + n\}$$

Conversely, one can determine n and $i_0, i_1, \ldots, i_\nu$ from D:

$$\begin{aligned} n &= \min(D) \\ i_j &= \min\{d \in D : v_p(d) \leq j\} - n \end{aligned}$$

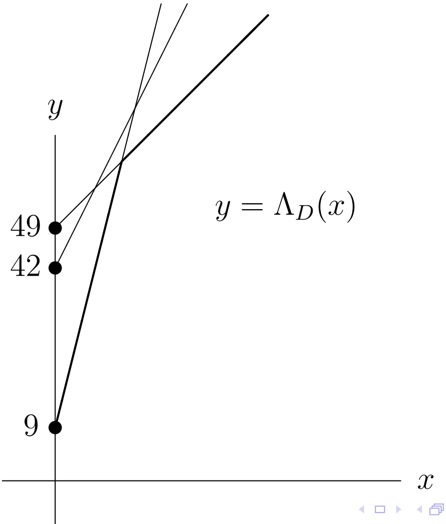
Let $\phi_{E/F}$ be the usual Hasse-Herbrand function. Then

$$\Lambda_D(r) = n \cdot \phi_{E/F}(r) + n.$$

An Example

Let $p = 3$ and $g(T) = T^9 + T^{36} - T^{42} + T^{48} - T^{49} + \cdots$.

Then $D = \mathrm{Ram}(g) = \{9, 42, 49\}$.



Some Sets

Let $D \subset \mathbb{N}$ be valid ramification data. Also let

$$\begin{aligned} D &= \{d_1, d_2, \ldots, d_m\} \text{ with } n = d_1 < d_2 < \cdots < d_m \\ Z &= \{i \in \mathbb{N} : \exists j\ i < d_j \ \& \ v_p(i) \leq v_p(d_j)\} \\ I &= \{\Lambda_D(c) : c \in \mathbb{N}\} \\ L &= \mathbb{N} \smallsetminus (D \cup Z \cup I) \end{aligned}$$

Then $\mathrm{Ram}(g) = D$ if and only if $a_i \neq 0$ for $i \in D$ and $a_i = 0$ for $i \in Z$.

By replacing $g(T)$ with $g(\gamma(T))$ for some $\gamma \in \mathcal{A}(k)$ we can assume $a_{d_1} = 1$ and $a_i = 0$ for $i \in I$.

Parameter Space

Let D be valid ramification data and set $D_0 = D \setminus \{n\}$.

Define a subset of $S(D)$ by

$$V(D) = \left\{ T^n + \sum_{i \in D_0 \cup L} a_i T^i : a_i \neq 0 \text{ for } i \in D_0 \right\}.$$

Then $V(D) \cong \prod_{i \in D_0} k^\times \times \prod_{i \in L} k$.

Let $\mathcal{E}(D)$ denote the set of isomorphism classes of finite extensions of F with ramification data D.

Define $\Theta_D : V(D) \to \mathcal{E}(D)$ by $\Theta_D(g) = \mathrm{Ext}(g)$.

Since every orbit of the action of $\mathcal{A}(k)$ on $S(D)$ is represented in $V(D)$, Θ_D is onto.

How Big are the Fibers?

Let $D = \{d_1, d_2, \ldots, d_m\}$ be valid ramification data.

Write $d_1 = n = up^\nu$ with $p \nmid u$.

Let $(x_1, y_1), \ldots, (x_t, y_t)$ be the vertices of the graph of Λ_D.

Define

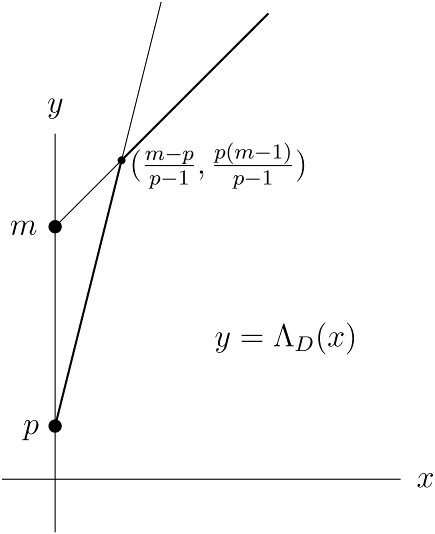
$$J(D) = u \cdot \prod_{x_i \in \mathbb{N}} \frac{\Lambda_D'(x_i - \epsilon)}{\Lambda_D'(x_i + \epsilon)}.$$

Theorem: Let E/F be an extension with ramification data D. Then the number of $g \in V(D)$ such that $\mathrm{Ext}(g) \cong E$ is $J(D)/|\mathrm{Aut}(E/F)|$.

Hence, for fixed ramification data D, the size of the fibers can vary.

Separable Extensions of Degree p

Let $D = \{p, m\}$, with $p < m$ and $p \nmid m$.



Extensions of Degree p, continued

$$
\begin{aligned}
D &= \{p, m\} \\
Z &= \{i < m : p \nmid i\} \\
I &= \{i > p : p \mid i \text{ if } i < \tfrac{p(m-1)}{p-1}\} \\
L &= \{i : m < i < \tfrac{p(m-1)}{p-1} \text{ and } p \nmid i\}
\end{aligned}
$$

Set $d = |L|$. Then $V(D) \cong k^\times \times k^d$.

Let $E \in \mathcal{E}(D)$. Then $[E : F] = p$, and

$$
J(D) = |\mathrm{Aut}(E/F)| = \begin{cases} p & \text{if } p-1 \mid m-1 \\ 1 & \text{if } p-1 \nmid m-1. \end{cases}
$$

The fibers of Θ_D have cardinality 1 in both cases. Therefore Θ_D is a bijection.

The Case $D = \{p, 2p-1\}$

$\mathcal{E}(D)$ is the set of isomorphism classes of cyclic extensions of F of degree p with ramification break 1.

Therefore elements of $\mathcal{E}(D)$ are generated by roots of Artin-Schreier equations

$$T^p - T - b^p x^{-1} = 0,$$

with $b \in k^\times$.

In this case $D_0 = \{2p-1\}$ and $L = \{\,\}$, so $V(D) \cong k^\times$.

The element of $V(D)$ which corresponds to the Artin-Schreier equation above is

$$g(T) = T^p + b^{1-p} T^{2p-1}.$$

The Case $D = \{p, 3p-2\}$, $p > 2$

$\mathcal{E}(D)$ is the set of isomorphism classes of cyclic extensions of F of degree p with ramification break 2.

Therefore elements of $\mathcal{E}(D)$ are generated by roots of Artin-Schreier equations

$$T^p - T - (b^p x^{-1} + c^p x^{-2}) = 0,$$

with $b \in k$, $c \in k^\times$.

In this case $D_0 = \{3p-2\}$ and $L = \{3p-1\}$, so $V(D) \cong k^\times \times k$.

The element of $V(D)$ which corresponds to the Artin-Schreier equation above is

$$g(T) = T^p + \tfrac{1}{2}c^{1-p}T^{3p-2} + \tfrac{1}{2}bc^{-p}T^{3p-1}.$$

The Case $D = \{2p, 4p-2\}$, $p > 2$

$\mathcal{E}(D)$ is the set of isomorphism classes of extensions of F of degree $2p$ with lower ramification breaks at 0 and 2.

Elements of $\mathcal{E}(D)$ are cyclic extensions of $F(x^{1/2})$ of degree p with ramification break 2.

Therefore elements of $\mathcal{E}(D)$ are generated over $F(x^{1/2})$ by roots of Artin-Schreier equations

$$T^p - T - (b^p x^{-1/2} + c^p x^{-1}) = 0,$$

with $b \in k$, $c \in k^\times$.

In this case $D_0 = \{4p-2\}$ and $L = \{4p-1\}$, so $V(D) \cong k^\times \times k$.

$D = \{2p, 4p-2\}$, continued

The element of $V(D)$ which corresponds to the Artin-Schreier equation above is

$$g(T) = T^{2p} + c^{1-p}T^{4p-2} + bc^{-p}T^{4p-1}.$$

$E = \mathrm{Ext}(g)$ is Galois over F if and only if $x^{1/2} \mapsto -x^{1/2}$ carries the Artin-Schreier equation to another which corresponds to the same p-extension.

This is equivalent to $a_{4p-1} = bc^{-p} = 0$, hence to $b = 0$.

Thus $J(D) = 2p$ and $|\mathrm{Aut}(E/F)| = 2p$ or p, depending on whether $a_{4p-1} = 0$ or not.

If E/F is Galois there is a unique $g \in V(D)$ with $\Theta_D(g) = E$. If E/F is not Galois then there are two such g.

A Nice Case

Suppose D is a set of valid ramification data such that

- $D \not\subset p\mathbb{N}$
- $n = p^\nu$ for some $\nu \geq 1$
- Λ_D has a single vertex (x_1, y_1)
- $x_1 \in \mathbb{N}$

Then every extension E/F with $\mathrm{Ram}(E/F) = D$ is Galois, with $\mathrm{Gal}(E/F)$ an elementary abelian p-group of order p^ν.

It follows that $\Theta_D : V(D) \to \mathcal{E}(D)$ is a bijection in this case.

How Useful are these Parametrizations?

$$\Theta_D : V(D) \longrightarrow \mathcal{E}(D)$$

- The construction only works for k algebraically closed.
- The construction can probably be extended to characteristic 0, but would be messier.
- Θ_D is onto, but is not a bijection in general. In fact, fibers can have different cardinalities.
- Galois and non-Galois extensions are parametrized by the same variety.
- How natural/canonical is this construction?