Hopf Galois Structures on Modular Extensions

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Outline

Let L/K be a finite extension.

Let H be a (finite, commutative, cocommutative) K-Hopf algebra, comultiplication Δ, counit ε, antipode λ.

Suppose L is an H-module and the action satisfies, for $h \in H, x, y \in L, \otimes = \otimes_K$:

$$\begin{aligned} h(1) &= \varepsilon(h)(1) \\ h(xy) &= \operatorname{mult} \Delta(h)(x \otimes y). \end{aligned}$$

Then L is an H-module algebra.

If, in addition, the map

$$\begin{aligned} L \otimes H &\to \operatorname{End}_K(L) \\ x \otimes h &\mapsto (y \mapsto xh(y)) \end{aligned}$$

is an isomorphism, then L/K is called Hopf Galois with Hopf algebra H, or H-Galois.

Example ("Hopf-Galois" $\supseteq$ "Galois")

If L/K is Galois, it is H-Galois, $H = KG, G = \mathrm{Gal}(L/K)$.
Action: $(a\sigma)(x) = a\sigma(x), a \in K, \sigma \in G, x \in L$.

Example ("Hopf-Galois" $\neq$ "Galois")

Let $K = \mathbb{Q}, L = \mathbb{Q}(\omega), \omega = \sqrt[3]{2}$. Then L/K is not Galois.
Let:

$$H = \mathbb{Q}[c,s]/(3s^2 + c^2 - 1, (2c+1)s, (2c+1)(c-1))$$

$$\Delta(c) = c \otimes c - 3s \otimes s, \varepsilon(c) = 1, \lambda(c) = c$$

$$\Delta(s) = c \otimes s + s \otimes c, \varepsilon(s) = 0, \lambda(s) = -s$$

$$\begin{array}{lll} c(1) = 1 & c(\omega) = -\frac{1}{2}\omega & c(\omega^2) = -\frac{1}{2}\omega^2 \\ s(1) = 0 & s(\omega) = \frac{1}{2}\omega & s(\omega^2) = -\frac{1}{2}\omega^2 \end{array}$$

Then L/K is H-Galois.

Suppose L/K is separable. Is L/K Hopf Galois?

If so, is it Hopf Galois for more than one Hopf algebra?

[Greither-Pareigis, 1987] reduces these to group theory questions.

Let E be a Galois closure of L/K, $G = \text{Gal}(E/K)$, and $G' = \text{Gal}(E/L)$.

There is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Hopf Galois structures} \\ \text{on } L/K \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{Regular subgroups of Perm}(G/G') \\ \text{(left cosets) normalized by } G \end{array} \right\}$$

So:

- Answers to existence/uniqueness questions only depend on the groups.
- The number of Hopf algebra structures on L/K is finite.

Suppose L/K is a primitive, purely inseparable extension, $[L:K] = p^n$.

Question. Is L/K Hopf Galois?

Yes [Chase, 1976].

Write $L = K(x), x^{p^n} = b \in K$.

Let $H = K[t]/(t^{p^n}), \Delta(t) = t \otimes 1 + 1 \otimes t, \varepsilon(t) = 0, \lambda(t) = -t$.

Define $\alpha : L \to L \otimes H$ by

$$\alpha(x) = x \otimes 1 + 1 \otimes t.$$

Taking duals gives an action $L \otimes H^* \to L$ which makes L/K H^*-Galois.

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Suppose L/K is a primitive, purely inseparable extension, $[L : K] = p^n$.

New Question. Is L/K Hopf Galois for more than one choice of Hopf algebra?

[Chase, 1976] suggests the answer is yes if $n > 1$.

We will show this is the case, and construct an example where L/K has an infinite number of Hopf Galois structures.

As with [Chase, 1976], we will consider the coaction dual to the Hopf Galois action.

Let H be a K-Hopf algebra. L is an H-Galois object if there exists a K-algebra map $\alpha : L \to L \otimes H$ such that

$$(\alpha \otimes 1)\alpha = (1 \otimes \Delta)\alpha$$
$$\text{mult}(1 \otimes \varepsilon)\alpha = 1$$

and the map $\gamma : L \otimes L \to L \otimes H$ given by

$$\gamma(x \otimes y) = x\alpha(y)$$

is an isomorphism.

Fact. L/K is H-Galois iff L is an H^*-Galois object.

Outline

Some Witt polynomials

For each $d \geq 0$, define $w_d(Z_0, \ldots, Z_d) \in \mathbb{Z}[Z_0, \ldots, Z_d]$ by

$$w_d(Z_0, \ldots, Z_d) = Z_0^{p^d} + pZ_1^{p^{d-1}} + \cdots + p^d Z_d.$$

Define $S_d \in \mathbb{Z}[X_0, \ldots, X_d, Y_0, \ldots, Y_d]$ recursively by

$$w_d(S_0, \ldots, S_d) = w_d(X_0, \ldots, X_d) + w_d(Y_0, \ldots, Y_d).$$

Yes, the coefficients are in $\mathbb{Z}$.

We continue to assume char $K = p$.
Pick $0 < r < n, f \in K^\times$. Let $d = \lceil n/r \rceil - 1$.
Define a sequence $\{f_i : i \geq 1\}$ in the algebraic closure $\bar{K}$ by

$$f_1 = f^{p^{-1}}, f_i = f^{p^{-i}} f_{i-1}^{p^r}, i > 1.$$

Let $H_{n,r,f} = K[t]/(t^{p^n})$ and define

$$\begin{aligned}
\Delta(t) &= S_d((f_d t^{p^{dr}} \otimes 1, \ldots, f_1 t^{p^r} \otimes 1, t \otimes 1); \\
&\qquad (1 \otimes f_d t^{p^{dr}}, \ldots, 1 \otimes f_1 t^{p^r}, 1 \otimes t)) \\
\varepsilon(t) &= 0 \\
\lambda(t) &= -t.
\end{aligned}$$

Proposition

$H_{n,r,f} = K[t]/(t^{p^n})$ is a K-Hopf algebra.

$H_{n,r,f} = K[t]/(t^{p^n}),$

$$\Delta(t) = S_d((f_d t^{p^{dr}} \otimes 1, \ldots, f_1 t^{p^r} \otimes 1, t \otimes 1);$$
$$(1 \otimes f_d t^{p^{dr}}, \ldots, 1 \otimes f_1 t^{p^r}, 1 \otimes t)).$$

Facts

1. $H_{n,r,f}$ is a local K-algebra.
2. $H^*_{n,r,f}$ is a local K-algebra.
   We say $H_{n,r,f}$ is “local-local".
3. If $f, f' \in K^\times$ and $f/f' \in (K^\times)^{p^{r+1}-1}$, then $H_{n,r,f} \cong H_{n,r,f'}$.
4. If $2r \geq n$ then the converse holds: $H_{n,r,f} \cong H_{n,r,f'}$ iff $f/f' \in (K^\times)^{p^{r+1}-1}$.

$$H_{n,r,f} = K[t]/(t^{p^n}),$$

$$\Delta(t) = S_d((f_d t^{p^{dr}} \otimes 1, \ldots, f_1 t^{p^r} \otimes 1, t \otimes 1); (1 \otimes f_d t^{p^{dr}}, \ldots, 1 \otimes f_1 t^{p^r}, 1 \otimes t)),$$

$$H_{n,r,f} \cong H_{n,r,f'} \text{ iff } f/f' \in (K^\times)^{p^{r+1}-1}.$$

Example

Let k be a perfect field, $K = k(T_1, T_2, \ldots)$. Let $2r \geq n$.
Then for all $i \neq j$, $H_{n,r,T_i} \ncong H_{n,r,T_j}$, and so there are an infinite number of isomorphism classes of these Hopf algebras.

Example (Case $d = 1$, i.e., $2r \geq n$.)

$$\begin{aligned}
\Delta(t) &= S_1((f_1 t^{p^r} \otimes 1, t \otimes 1); (1 \otimes f_1 t^{p^r}, 1 \otimes t)) \\
&= t \otimes 1 + 1 \otimes t + \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} (f_1 t^{p^r})^\ell \otimes (f_1 t^{p^r})^{p-\ell} \\
&= t \otimes 1 + 1 \otimes t + f_1^p \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} t^{p^r \ell} \otimes t^{p^r (p-\ell)} \\
&= t \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} t^{p^r \ell} \otimes t^{p^r (p-\ell)}.
\end{aligned}$$

(Recall $f_1 = f^{p^{-1}}$.)

Outline

Let $L = K(x), x^{p^n} = b \in K, [L : K] = p^n$.

Define $\alpha : L \to L \otimes H_{n,r,f}$ by

$$\alpha(x) = S_d((f_d x^{p^{dr}} \otimes 1, \ldots, f_1 x^{p^r} \otimes 1, x \otimes 1); (1 \otimes f_d t^{p^{dr}}, \ldots, 1 \otimes f_1 t^{p^r}, 1 \otimes t)).$$

(Can show $\alpha(x)^{p^n} = b \otimes 1$, so α is a well-defined K-algebra map.)

Since Witt vector addition is associative, $(\alpha \otimes 1)\alpha = (1 \otimes \Delta)\alpha$.

Also, since $\varepsilon(t) = 0$,

$$\begin{aligned}(1 \otimes \varepsilon)\alpha(x) &= S_d((f_d x^{p^{dr}} \otimes 1, \ldots, x \otimes 1); (1 \otimes f_d \varepsilon(t)^{p^{dr}}, \ldots, 1 \otimes \varepsilon(t))) \\ &= x \otimes 1,\end{aligned}$$

so $\mathrm{mult}(1 \otimes \varepsilon)\alpha = 1$.

The induced map $\gamma : L \otimes L \to L \otimes H_{n,r,f}$ is an isomorphism.

Thus, L is an $H_{n,r,f}$-Galois object.

The coaction depends on the chosen algebra generator of L.

Example (Assume p odd.)

Let $y = x^2$. Then $L = K(y)$ and $x = b^{-1}y^{(p^n+1)/2}$. Then we can define $\alpha' : L \to L \otimes H_{n,r,f}$ by

$$\alpha'(y) = S_d((f_d y^{p^{dr}} \otimes 1, \dots, f_1 y^{p^r} \otimes 1, y \otimes 1); (1 \otimes f_d t^{p^{dr}}, \dots, 1 \otimes f_1 t^{p^r}, 1 \otimes t)).$$

So

$$\alpha'(x) = b^{-1}\Delta(y)^{(p^n+1/2)} \neq \alpha(x).$$

More generally, let $y_i = x^i$, $1 \leq i \leq p^n - 1, p \nmid i$.
Then $L = K(y_i)$, and

$$\alpha_i(y_i) = S_d((f_d y_i^{p^{dr}} \otimes 1, \dots, f_1 y_i^{p^r} \otimes 1, y_i \otimes 1); (1 \otimes f_d t^{p^{dr}}, \dots, 1 \otimes f_1 t^{p^r}, 1 \otimes t))$$

can be used to make L an $H_{n,r,f}$-Galois object.

We have $p^{n-1}(p-1)$ different coactions.

Theorem

Let $[L:K]=p^n, L=K(x), x^{p^n}\in K$*. For* $0<r<n$ *and* $f\in K^\times$*, let* $H_{n,r,f}=K[t]/(t^{p^n})$,

$$\Delta(t)=S_d((f_d t^{p^{dr}}\otimes 1,\dots,f_1 t^{p^r}\otimes 1, t\otimes 1);(1\otimes f_d t^{p^{dr}},\dots,1\otimes f_1 t^{p^r},1\otimes t)).$$

Then L/K *is* $H^*_{n,r,f}$*-Galois.*

If we allow $n=r$ then $H_{n,n,0}$ is Chase's Hopf algebra (and the choice of f is irrelevant).

Outline

The Greither-Pareigis results describe actions, not coactions.

Suppose $r = n - 1$.

Then $H_{n,n-1,f} \cong H_{n,n-1,f'}$ iff $f/f' \in (K^\times)^{p^{r+1}-1}$.

Let $H = H^*_{n,n-1,f}$.

Objectives.

1. Explicitly describe the Hopf algebra structure on H.
2. Explicitly describe an action of H on L.

Let $H = H^*_{n,n-1,f}$

H has K-basis $\{z_j : 0 \leq j \leq p^n - 1\}$, where

$$z_j(t^i) = \delta_{i,j} \text{ (Kronecker delta).}$$

The algebra structure arises from the coalgebra structure on $H_{n,n-1,f}$:

$$\begin{aligned}
(z_{j_1} z_{j_2})(t^i) &= \operatorname{mult}(z_{j_1} \otimes z_{j_2})\Delta(t^i) \\
&= \operatorname{mult}(z_{j_1} \otimes z_{j_2})(S_1((f_1 t^{p^{n-1}} \otimes 1, t \otimes 1); (1 \otimes f_1 t^{p^{n-1}}, 1 \otimes t))^i \\
&= \operatorname{mult}(z_{j_1} \otimes z_{j_2})\left(t \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} t^{p^{n-1}\ell} \otimes t^{p^{n-1}(p-\ell)}\right)^i.
\end{aligned}$$

$$(z_{j_1} z_{j_2})(t^i) = \mathrm{mult}(z_{j_1} \otimes z_{j_2})\left(t \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} t^{p^{n-1}\ell} \otimes t^{p^{n-1}(p-\ell)}\right)^i$$

For for $1 \leq m \leq p-1, 0 \leq s \leq n-1$ it can be shown:[1]

- $z_{p^s}^m = m! z_{mp^s}$
- for $s \neq n-1, z_{p^s}^p = 0$
- $z_{p^{n-1}}^p = f z_1$
- $z_{p^{n-1}}^{p^2} = 0$
- H is generated as a K-algebra by $\{z_p, \ldots, z_{p^{n-1}}\}$:

$$H = K[z_p, \ldots, z_{p^{n-1}}]/(z_p^p, \ldots, z_{p^{n-2}}^p, z_{p^{n-1}}^{p^2})$$

[1] More detail Thursday.

The comultiplication on H is induced from the multiplication on $H_{n,n-1,f}$:

$$\Delta(z_j) \in H \otimes H \cong \mathrm{Hom}_K(H_{n,n-1,f} \otimes H_{n,n-1,f}, K)$$
$$\Delta(z_j)(t^{i_1} \otimes t^{i_2}) = z_j(t^{i_1+i_2}) = \delta_{(i_1+i_2),j}.$$

So

$$\Delta(z_j) = \sum_{i=0}^{j} z_i \otimes z_{j-i},$$

which can be written in terms of the algebra generators $\{z_{p^s}\}$ using p-adic expansions for each i.

In particular[2], since $z_1^i = i! z_i$ for $i < p$ and $z_{p^{n-1}}^p = f z_1$,

$$\begin{aligned}
\Delta(z_p) &= \sum_{i=0}^{p} z_i \otimes z_{p-i} \\
&= z_p \otimes 1 + 1 \otimes z_p + \sum_{i=1}^{p-1} z_i \otimes z_{p-i} \\
&= z_p \otimes 1 + 1 \otimes z_p + \sum_{i=1}^{p-1} (i!)^{-1} z_1^i \otimes ((p-i)!)^{-1} z_1^{p-i} \\
&= z_p \otimes 1 + 1 \otimes z_p + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} (f^{-1} z_{p^{n-1}}^p)^i \otimes (f^{-1} z_{p^{n-1}}^p)^{p-i} \\
&= z_p \otimes 1 + 1 \otimes z_p + f^{-p} \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} z_{p^{n-1}}^{pi} \otimes z_{p^{n-1}}^{p(p-i)} \\
&= S_1((f^{-1} z_{p^{n-1}} \otimes 1, z_p \otimes 1); (1 \otimes f^{-1} z_{p^{n-1}}, 1 \otimes z_p)).
\end{aligned}$$

[2] Less detail Thursday.

The Hopf Galois action $L \otimes H \to L$ is induced by the coaction $\alpha : L \to L \otimes H_{n,n-1,f}$:

$$z_{p^s}(x^i) = \mathrm{mult}(1 \otimes z_{p^s})\alpha(x^i).$$

Explicitly, if $i = \sum_{\ell=0}^{n-1} i_\ell p^\ell$ then:

$$z_{p^s}(x^i) = i_s x^{i-p^s}, 0 \leq s < n-1$$
$$z_{p^{n-1}}(x^i) = i_{n-1} x^{i-p^{n-1}} - ifbx^{i-p^{n-1}-1}.$$

(Recall $b = x^{p^n} \in K$.)

Example

Let k be a perfect field.
$K = k(T_1, T_2, \ldots)$.
$L = K(x), x^{p^n} = b \in K, [L : K] = p^n, n > 1$.
Then for all $i \neq j, H_{n,n-1,T_i} \not\cong H_{n,n-1,T_j}$ as before.
L/K is H-Galois for an infinite number of Hopf algebras H.

Contrast to the L/K separable case:

1. Finite number of Hopf algebras.
2. Exact number independent of K.

Outline

Now suppose L/K is modular, i.e.

$$L \cong K(x_1) \otimes \cdots \otimes K(x_n), [K(x_i) : K] = p^{e_i}$$

[Chase, 1976] shows L is an H-Galois object for

$$H = K[t_1, \ldots, t_n]/(t_1^{p^{e_1}}, \ldots, t_n^{p^{e_n}}), \Delta(t_i) = t_i \otimes 1 + 1 \otimes t_i.$$

This is a tensor product of Chase's Hopf algebras $H_{e_i,e_i,0}$.

Let

$$H = H_{e_1,r_1,f_1} \otimes \cdots \otimes H_{e_n,r_n,f_n}, 0 < r_n < e_n.$$

Then L can be made into an H-Galois object in

$$p^{e_1-1}p^{e_2-1}\cdots p^{e_n-1}(p-1)^n = [L:K]p^{-n}(p-1)^n$$

different ways.

Is L an H-Galois object for a more interesting choice of H?

Example (1)

Let $L = K(x, y), [K(x) : K] = [K(y) : K] = p^2$.
Let $H = K(t, u)/(t^{p^2}, u^{p^2})$ with

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{\ell=1}^{p-1} * \, u^{p\ell} \otimes u^{p(p-\ell)}$$

$$\Delta(u) = u \otimes 1 + 1 \otimes u.$$

Define

$$\alpha(x) = x \otimes 1 + 1 \otimes t + \sum_{\ell=1}^{p-1} * \, y^{p\ell} \otimes u^{p(p-\ell)}$$

$$\alpha(y) = y \otimes 1 + 1 \otimes u.$$

Then L is an H-Galois object.

$$* = \frac{1}{\ell!\,(p-\ell)!}.$$

Example (2)

Same $L = K(x,y)$. Let $H = K(t,u)/(t^{p^2}, u^{p^2})$ with

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{\ell=1}^{p-1} (t-u)^{p\ell} \otimes (t-u)^{p(p-\ell)}$$

$$\Delta(u) = u \otimes 1 + 1 \otimes u + \sum_{\ell=1}^{p-1} u^{p\ell} \otimes u^{p(p-\ell)}.$$

Define

$$\alpha(x) = x \otimes 1 + 1 \otimes t + \sum_{\ell=1}^{p-1} (x-y)^{p\ell} \otimes (t-u)^{p(p-\ell)}$$

$$\alpha(y) = y \otimes 1 + 1 \otimes u + \sum_{\ell=1}^{p-1} y^{p\ell} \otimes u^{p(p-\ell)}.$$

Then L is an H-Galois object.

Crazy Idea

Let L/K be modular, and let H be a local-local Hopf algebra such that

$$L \otimes L \cong L \otimes H^*$$

as K-algebras. Then L is an H^*-Galois object (and hence L/K is H-Galois).

In other words, whether L/K is H-Galois depends only on the algebra structure of H^*.

Loosely,

$$L \cong K(x_1) \otimes \cdots \otimes K(x_n), [K(x_i) : K] = p^{e_i}$$
$$H = K(t_1, \ldots, t_n)/(t_1^{p^{e_1}}, \ldots, t_n^{p^{e_n}}),$$

and $\alpha : L \to L \otimes H$ arises from substituting x_i for t_i in the first factor of $\Delta(t_i)$ for all i.

Outline

1. Overview
2. A Construction
3. Primitive Extensions
4. Example: The Case $r = n - 1$
5. Modular Extensions
6. What's Next

Work to come (hopefully):

- Explicit computations of $H^*_{n,r,f}$ and its action on L when $r < n-1$, especially $r < n/2$.
- A necessary condition for $H_{n,r,f} \cong H_{n,r,f'}$.
  - Conjecture: the $2r \geq n$ condition extends to all r.
- For a fixed K, a count (or parameterization) of the isomorphism classes of Hopf algebras $H_{n,r,f}$.
- More examples not of the form $\bigotimes H_{n_i,r_i,f_i}$.
- Suppose k finite, $K = k((T)), \mathfrak{O}_K = k[[T]]$.
  - A classification of $\mathfrak{O}_K$-Hopf orders in $H_{n,r,f}$
  - A classification of $\mathfrak{O}_K$-Hopf orders in $H^*_{n,r,f}$
- Proof (or counterexample) to the Crazy Idea (but not right away).

Thank you.