Vortrag in Exeter am 22. Juni 2015

Euler characteristics and ε-constants of curves over finite fields

Some wild stuff

(1)

(with Helena Fischbacher-Weitz).

X (geometrically) irreducible smooth projective curve over $\mathbb{F}_p$ of genus g_X.

= (think: function field of transcdegree 1 over $\mathbb{F}_p$)

$$Z(X,t) := \prod_{y \in X} \frac{1}{1-t^{\deg(y)}} \qquad (t = p^{-s}, \ \deg(y) := [k(y) : \mathbb{F}_p]).$$

$$= \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{p^r})| \frac{t^r}{r} \right)$$

e.g. $X = \mathbb{P}^1_{\mathbb{F}_p} \Rightarrow Z(X,t) = \exp\left(\sum_{r=1}^{\infty} (p^r+1) \frac{t^r}{r} \right) = \frac{1}{(1-t)(1-pt)}$

Functional equation: $Z(X,t) = p^{\boxed{1-g_X}} \, t^{?} \, Z(X, \frac{1}{pt})$

$G \subseteq \mathrm{Aut}(X)$ finite. For $y \in X$ let

$\cup$

$G_y := \{ \sigma \in G : \sigma(y) = y \}$

$\cup$

$I_y := \{ \sigma \in G_y : \sigma$ induces id on $k(y) = \mathcal{O}_{X,y} / m_y \mathcal{O}_{X,y} \}$

$\cup$

$G_{y,1} := \{$ ——————— " ——— $\mathcal{O}_{X,y} / m_y^2 \mathcal{O}_{X,y} \}$

$\cup$

$\vdots$

$\pi : X \to X/G =: Y$

$G_y / I_y \xrightarrow{\sim} \mathrm{Aut}(k(y) / k(\pi(y)))$

$\mathrm{Frob}(y) \mapsto (x \mapsto x^{|k(\pi(y))|})$

For f.d. complex representation V of G define

Artin L-function $L(V,t) := \prod_{y \in X} \left(1 - \mathrm{Frob}(y) \, t^{\deg(y)} \mid V^{I_y} \right)^{-1}$

Functional equation: $L(V,t) = \varepsilon(V) \, t^{?} \, L(V^*, \frac{1}{pt})$

Fact: ε-constant $\varepsilon(V) \in \bar{\mathbb{Q}}^{\times}$.

Fix: $j_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$; $v_p : \bar{\mathbb{Q}}_p \longrightarrow \mathbb{Q}$ p-adic valuation 2

$\langle \ , \ \rangle$ classical (character) pairing on

$$\begin{tikzcd} K_0(\mathbb{C}[G]) & K_0(\bar{\mathbb{Q}}[G]) \arrow[l, "\sim"'] \arrow[r, "j_p"] & K_0(\bar{\mathbb{Q}}_p[G]) \arrow[dr, "d"] & \\ & & K_0(\bar{\mathbb{F}}_p[G]) \arrow[u, "e"] \arrow[r, hook, "c"] & K_0(G, \bar{\mathbb{F}}_p) \end{tikzcd} \quad (= G_0(\bar{\mathbb{F}}_p[G]))$$

Define $E(G,X) \in K_0(\bar{\mathbb{Q}}_p[G])_{\mathbb{Q}}$ by:

$$\langle E(G,X), j_p(V) \rangle = v_p(j_p(\varepsilon(V))) \text{ in } \mathbb{Q}. \quad \forall V \text{ as above.}$$

$\bar{X} := X \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$

Equivariant Euler characteristic:

$$\chi(G, \bar{X}, \mathcal{O}_{\bar{X}}) := \underset{\bar{\mathbb{F}}_p}{\underset{\parallel}{[H^0(\bar{X}, \mathcal{O}_{\bar{X}})]}} - \underset{H^0(\bar{X}, \Omega_{\bar{X}})^*}{\underset{\wr\parallel}{[H^1(\bar{X}, \mathcal{O}_{\bar{X}})]}} \in K_0(G, \bar{\mathbb{F}}_p)$$

'Weak' Theorem: $\boxed{d(E(G,X)) = \chi(G, \bar{X}, \mathcal{O}_{\bar{X}})}$ in $K_0(G, \bar{\mathbb{F}}_p)$

Pf. . . Thm of Chinburg (1994, Annals)

· Artin's induction thm for modular representation theory.

Now: $\pi : X \longrightarrow Y$ weakly ramified, i.e. $G_{y,2} = \{1\}$ $\forall y \in X$.

$X^w := \{ y \in X : G_{y,1} \neq \{1\} \}$, $D^w := - \sum_{y \in X^w} [y]$

$\exists!\ \psi(G, \bar{X}) \in K_0(\bar{\mathbb{F}}_p[G])$ s.t.

$$c(\psi(G,\bar{X})) = [H^0(\bar{X}, \mathcal{O}_{\bar{X}}(D^w))] - [H^1(\bar{X}, \mathcal{O}_{\bar{X}}(D^w))] \text{ in } K_0(G, \bar{\mathbb{F}}_p)$$

(from my paper in Amer. J. Math.)

For every $\bar{Q} \in \bar{Y}^W = \bar{\pi}(\bar{X}^W)$ choose $\tilde{Q} \in \pi^{-1}(Q)$.

Main Theorem $\boxed{E(G,X) = e(\psi(G,\bar{X})) + \sum_{Q \in \bar{Y}^W} [\mathrm{Ind}_{G_{\tilde{Q}}}^{G}(1)]^{-1}}$ in $K_0(\bar{\mathbb{Q}}_p[G])$

In particular: $E(G,X) \in K_0(\bar{\mathbb{Q}}_p[G])$.

About the proof. By Artin's induction theorem it suffices to consider G abelian and V corresponds to a multiplicative character χ.

$y \in Y^t := \{y \in Y : \pi \text{ tamely ramified above } y\}$.

$$\begin{tikzcd}
\widehat{K(Y)}_y \arrow[r, "\text{norm residue}", "\text{homom}"'] & G_{\tilde{y}} \subseteq G \arrow[r, "\chi"] & \bar{\mathbb{Q}}^\times \subseteq \mathbb{C}^\times \\
\hat{\mathcal{O}}^\times_{Y,y} \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d] & I_{\tilde{y}} \arrow[u, phantom, "\cup"] & \\
k(y) \arrow[ur] \arrow[uurr, "\chi_{k(y)}"'] & &
\end{tikzcd}$$

Gauss sum: $\sum_{x \in k(y)^\times} \chi_{k(y)}(x)\, \zeta_p^{\mathrm{Tr}_{k(y)/\mathbb{F}_p}(x)} =: \tau(\chi_{k(y)})$

Statement 1

$$\langle e(\psi(G,\bar{X})), j_p(\chi)\rangle = (1-g_Y) - \sum_{y \in Y^t} v_p(j_p(\tau(\chi_{k(y)}))) - \sum_{y \in Y^W} \deg(y).$$

Statement 2: u_p to a multiplicative root of unity:

$$\varepsilon(\chi^{-1}) = p^{g_Y - 1} \prod_{y \in Y^t(\chi)} \tau(\chi_{k(y)}) \prod_{y \in Y^W(\chi)} |k(y)|.$$

1. Based on my paper in Amer. J. Math.
   - + Stickelberger's formula for the p-adic valuation of Gauss sums
   - + class field theory
   - + 10 pages computations.

(2) Based on Deligne's description of $\varepsilon(X)$

- compute Tamagawa measure on $\mathbb{A}_{K(Y)}$ ("Mittag-Leffler").
- construct additive character ψ on $\mathbb{A}_{K(Y)}/K(Y)$ (uses residue theorem)
- compute local p-adic integrals.