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23/06/2015

Explicit integral Galois module structure of weakly ramified extensions of local fields

Thank members of audience, including:

Alex Bartel, Nigel Byott, Griff Elder, Cornelius Greiter, Bernhard Köck.

To appear in Proc AMS.

Setup: finite Galois extension of local fields

$$G \left(\begin{array}{c} L \\ | \\ K \end{array} \right) \quad \begin{array}{l} \mathcal{O}_L \supset \mathcal{P}_L \\ \mathcal{O}_K \supset \mathcal{P}_K \end{array} \quad \begin{array}{l} \bar{L} = \mathcal{O}_L / \mathcal{P}_L \\ \bar{K} = \mathcal{O}_K / \mathcal{P}_K \end{array} \quad \begin{array}{l} \text{Residue fields} \\ \text{assumed to be finite} \end{array}$$

Concerned with:

(i) \mathcal{P}_L^n as an $\mathcal{O}_K[G]$ -module

(ii) \mathcal{O}_L as an $\mathcal{U}_{L/K}$ -module $\mathcal{U}_{L/K} = \{x \in K[G] : x\mathcal{O}_L \subseteq \mathcal{O}_L\}$

Ramification groups For $i \geq -1$

$$G_i := \{g \in G : (g-1)(\mathcal{O}_L) \subseteq \mathcal{P}_L^{i+1}\}$$

Hence

$$L/K \text{ unramified} \Leftrightarrow G_0 = 1$$

$$L/K \text{ tamely ramified} \Leftrightarrow G_1 = 1$$

$$L/K \text{ weakly ramified} \Leftrightarrow G_2 = 1.$$

Tame case

$\forall n \in \mathbb{Z}$ \mathcal{P}_L^n free over $\mathcal{O}_K[G]$ (Neether*)

Kawamoto (1986) constructed explicit generators.

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Unramified case

$$G \left(\begin{array}{c} L \\ | \\ K \end{array} \right) \quad G \left(\begin{array}{c} \bar{L} \\ | \\ \bar{K} \end{array} \right)$$

By Normal Basis Theorem $\exists \bar{\beta} \in \bar{L}$ s.t. $\bar{L} = \bar{K}[G] \cdot \bar{\beta}$.

Using Nakayama's Lemma, for any lift $\beta \in \mathcal{O}_L$ of $\bar{\beta}$, $\mathcal{O}_L = \mathcal{O}_K[G] \cdot \beta$. (Can make into an iff statement.)

Totally and tamely ramified case

$$G \left(\begin{array}{c} L \\ | \\ K \end{array} \right)$$

Let $e = [L:K]$.

\exists uniformizers $\pi_L \in \mathcal{O}_L$ & $\pi_K \in \mathcal{O}_K$ s.t. $\pi_L^e = \pi_K$. $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$

Let $\alpha \in \mathcal{O}_L$. Then $\alpha = u_0 + u_1 \pi_L + \dots + u_{e-1} \pi_L^{e-1}$, $u_i \in \mathcal{O}_K$.

$$\mathcal{O}_L = \mathcal{O}_K[G] \cdot \alpha \iff u_i \in \mathcal{O}_K^\times \quad \forall i$$

(In particular, $\alpha = 1 + \pi_L + \pi_L^2 + \dots + \pi_L^{e-1}$ is a generator.)

Proof: Use that π_L is a Kummer generator, determinant calculation.

Idea: "G-we" the two cases together.

Weakly ramified case

Ullom:

- (i) If $\exists n \in \mathbb{Z}$ s.t. B_L^n free over $\mathcal{O}_K[G]$, then L/K weakly ramified
- (ii) If L/K totally & weakly ramified, then P_L free over $\mathcal{O}_K[G]$.

Köck B_L^n free over $\mathcal{O}_K[G] \iff L/K$ weakly ramified & $n \equiv 1 \pmod{|G|}$

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Proof uses cohomological triviality argument
(Erez's work on square root of inverse different uses similar ideas.)

Does not construct explicit generators.

Theorem 1 L/K weakly ramified. Let $n \in \mathbb{Z}$ st. $n \equiv 1 \pmod{|G|}$.
Then one can explicitly construct ϵ s.t. $P_L^n = \mathcal{O}_L[G] \cdot \epsilon$.

Theorem 2 L/K weakly ramified. π_K any uniformizer of K .
Then $\mathcal{U}_{L/K} = \mathcal{O}_K[\mathcal{O}] [\pi_K^{-1} \text{Tr}_{\mathcal{O}_0}]$ and if $P_L = \mathcal{O}_L[G] \cdot \epsilon$
then $\mathcal{O}_L = \mathcal{U}_{L/K} \cdot \epsilon$.

Idea of proof of Theorem 1

Explicitly construct generators in following cases:

- (i) unramified ✓
- (ii) totally & tamely ramified ✓
- (iii) totally & weakly ramified p -extension

Then use "splitting lemma".

"Glue" generators together.

Take trace.

This is a generalisation of Kawamoto's approach.

Totally & weakly ramified p -extension

Thm L/K totally & weakly ramified p -extension ($p = \text{char } K > 0$).

(i) G elementary abelian p -group (standard)

(ii) β_L^n free over $\mathcal{O}_K[\mathcal{O}] \Leftrightarrow n \equiv 1 \pmod{|G|}$ (denoted by K\"och)

(iii) Suppose $n \equiv 1 \pmod{|G|}$.

Then $\delta \in L$ free gen of β_L^n over $\mathcal{O}_K[\mathcal{O}]$

$\Leftrightarrow v_L(\delta) = n$.

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(iii) Already shown by others (sometimes with restrictions)
by Vostokov, Vinaker (Bjott), Bjott & Elder.

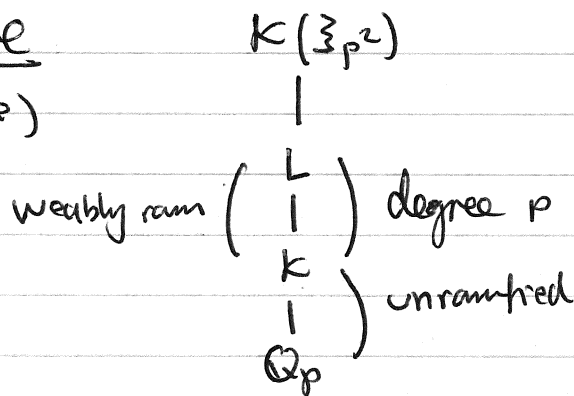
(In fact, works for perfect residue fields of the characteristic)

Proof Elementary

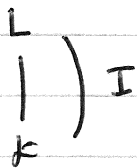
- Use Hilbert's formula to compute different of L/K .
- Obtain formula for $\text{Tr}_{L/K}(B_L^n)$.
- "Mod out" by $B_K \rightarrow$ work over $\bar{K}[G]$.
- Use (minor variant) of result of Childs
- Lift using Nakayama. □

Example

(if time)



Totally & weakly ramified extensions of arbitrary degree

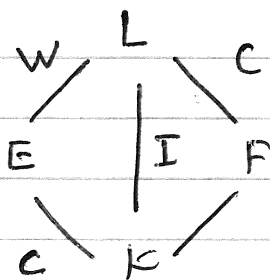


$I = W \times C$

by Schur-Zassenhaus

↑ cyclic

wild inertia $\cong G_1 =$ elementary abelian p -extension



$L/E, F/K$ totally weakly ram p -extensions
 $L/F, E/K$ totally and tamely ramified.

Define r by $|W| = p^r$, let $c = |C|$.
By Bézout $\exists a, b \in \mathbb{Z}$
st. $ap^r + bc = 1$.

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(Special case for simplicity)

Proposition π_P any uniformizer $\alpha := 1 + \pi_E + \pi_E^2 + \dots + \pi_E^{e-1}$.
Then $\pi_P^b \pi_E^a \cdot \alpha$ free gen of P_L over $O_K [I]$.

Proof

(i) $v_L(\pi_P^b \pi_E^a) = 1$ so $\beta_L = O_E [W] \cdot (\pi_P^b \pi_E^a)$

(ii) $\pi_E^a O_E = O_K [C] \cdot (\pi_E^a \alpha)$

Do explicit calculation using semidirect product and that $\pi_P \in F = L^C, \pi_E, \alpha \in E = L^W$.

(Optional) $W = \{\tau_i\} \quad C = \{\sigma_j\}$.

Starts like:

$$\begin{aligned}
P_L &= O_E [W] \cdot (\pi_P^b \pi_E^a) && \text{(i)} \\
&= \bigoplus_i \tau_i (\pi_P^b \pi_E^a) O_E \\
&= \bigoplus_i \tau_i (\pi_P^b) \cdot \pi_E^a O_E && \text{since } \pi_E \in E = L^W \\
&= \text{(use (ii)) } \dots \\
&= \dots \\
&= \bigoplus_i \bigoplus_j \tau_i \sigma_j (\pi_P^b \pi_E^a \cdot \alpha) \\
&= O_K [I] \cdot (\pi_P^b \cdot \pi_E^a \cdot \alpha).
\end{aligned}$$

Rmk If L/K abelian, totally & wildly ramified, not of p -power degree then L/K cannot be weakly ramified.

e.g.	$\mathbb{Q}_p(\zeta_{p^2})$) tame		But $\mathbb{Q}_3(\zeta_3, \sqrt[3]{2})/\mathbb{Q}_3$
not weakly ramified	$\downarrow \uparrow_{p-1}$			
	K) weakly ram		Galois group S_3
	$\downarrow \uparrow_p$			
	\mathbb{Q}_p			Totally & weakly ramified.

(type in paper)

Doubly split extensions

L/k finite Galois ext. of local fields.

$$G = \text{Gal}(L/k) \quad I = G_0 \quad W = G_1$$

We say L/k is:

- (i) split wrt inertia if $G = I \rtimes U$
for some (cyclic) U (so L/L^U unramified).
- (ii) split wrt wild inertia if $G = W \rtimes T$
for some T (so L/L^T tamely ramified).
- (iii) doubly split if $\exists C \leq I$ and $U \leq T$
 $I = W \rtimes C \quad T = C \rtimes U$

$$\text{so } G = W \rtimes T = W \rtimes (C \rtimes U) = (W \rtimes C) \rtimes U = I \rtimes U$$

Rank Automatic in totally ramified case by Schur Zassenhaus.

Give generators together for doubly split extension.

Lemma L/k finite Galois ext of local fields.

Let k'/k be unique unramified extension of degree $[L:k]$.

Let $L' = Lk'$. Then

- (i) L'/k Galois
- (ii) $\text{Gal}(L'/k')$ inertia subgroup
- (iii) L'/k doubly split
- (iv) if L/k weak ram then L'/k weak ram.

Proof Group Theory. □

For general L/k , construct gen ϵ' for L'/k .
Then $\epsilon := \text{Tr}_{L'/L}(\epsilon')$ is gen for L/k .