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Explicit integral Galois module structure of weakly ramified extensions of local fields

Thank members of audience, including:
Alex Bartel, Nigel Byott, Griff Elder, Cornelius Greither, Bernhard Köck.

To appear in Proc AMS.

Setup: finite Galois extension of local fields

$$\begin{array}{llll} & L & \mathcal{O}_L \rhd \mathcal{P}_L & \bar{L} = \mathcal{O}_L/\mathcal{P}_L \\ G & \mid & & \mid \\ & K & \mathcal{O}_K \rhd \mathcal{P}_K & \bar{K} = \mathcal{O}_K/\mathcal{P}_K \end{array}$$

Residue fields assumed to be finite

Concerned with:

(i) $\mathcal{P}_L^n$ as an $\mathcal{O}_K[G]$-module

(ii) $\mathcal{O}_L$ as an $\mathfrak{A}_{L/K}$-module $\mathfrak{A}_{L/K} = \{x \in K[G] : x\mathcal{O}_L \subseteq \mathcal{O}_L\}$

Ramification groups For $i \geqslant -1$

$$G_i := \{\sigma \in G : (\sigma - 1)(\mathcal{O}_L) \subseteq \mathcal{P}_L^{i+1}\}$$

Hence

L/K unramified $\Leftrightarrow G_0 = 1$

L/K tamely ramified $\Leftrightarrow G_1 = 1$

L/K weakly ramified $\Leftrightarrow G_2 = 1$.

Tame case

$\forall n \in \mathbb{Z}$ $\mathcal{P}_L^n$ free over $\mathcal{O}_K[G]$ (Noether*)

Kawamoto (1986) constructed explicit generators.

Unramified case

$$G\left(\begin{array}{c} L \\ | \\ K \end{array}\right. \qquad G\left(\begin{array}{c} \bar{L} \\ | \\ \bar{K} \end{array}\right.$$

By Normal Basis Theorem $\exists \bar{\beta} \in \bar{L}$ s.t. $\bar{L} = \bar{K}[G] \cdot \bar{\beta}$.
Using Nakayama's lemma, for any lift $\beta \in \mathcal{O}_L$ of $\bar{\beta}$, $\mathcal{O}_L = \mathcal{O}_K[G] \cdot \beta$. (Can make into an iff statement.)

Totally and tamely ramified case

$$G\left(\begin{array}{c} L \\ | \\ K \end{array}\right.$$

Let $e = [L:K]$.
$\exists$ uniformizers π_L & π_K s.t. $\pi_L^e = \pi_K$. $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$

Let $\alpha \in \mathcal{O}_L$. Then $\alpha = u_0 + u_1 \pi_L + \dots + u_{e-1} \pi_L^{e-1}$, $u_i \in \mathcal{O}_K$.

$$\mathcal{O}_L = \mathcal{O}_K[G] \cdot \alpha \iff u_i \in \mathcal{O}_K^\times \quad \forall i$$

(In particular, $\alpha = 1 + \pi_L + \pi_L^2 + \dots + \pi_L^{e-1}$ is a generator).

Proof: Use that π_L is a Kummer generator; determinant calculation $\square$

Idea: "Glue" the two cases together.

Weakly ramified case

Ullom:
(i) If $\exists n \in \mathbb{Z}$ s.t. $\mathfrak{P}_L^n$ free over $\mathcal{O}_K[G]$, then L/K weakly ramified
(ii) If L/K totally & weakly ramified, then $\mathfrak{P}_L$ free over $\mathcal{O}_K[G]$.

Köck $\mathfrak{P}_L^n$ free over $\mathcal{O}_K[G] \iff L/K$ weakly ramified & $n \equiv 1 \mod |G_1|$

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<u>Proof</u> Uses cohomological triviality argument.
(Erez's work on square root of inverse different uses similar ideas)

Does not construct explicit generators.

<u>Theorem 1</u> L/K weakly ramified. Let $n \in \mathbb{Z}$ s.t. $n \equiv 1 \mod |G|$.
Then one can explicitly construct $\mathcal{E}$ s.t. $P_L^n = \mathcal{O}_L[G] \cdot \mathcal{E}$.

<u>Theorem 2</u>. L/K weakly ramified. π_K any uniformizer of K.
Then $\mathfrak{A}_{L/K} = \mathcal{O}_K[G][\pi_K^{-1} Tr_{G_0}]$ and if $P_L = \mathcal{O}_L[G] \cdot \mathcal{E}$
then $\mathcal{O}_L = \mathfrak{A}_{L/K} \cdot \mathcal{E}$.

<u>Idea</u> of proof of Theorem 1
Explicitly construct generators in following cases:
- (i) unramified ✓
- (ii) totally & tamely ramified ✓
- (iii) totally & weakly ramified p-extension

Then use "splitting lemma".
"Glue" generators together.
Take trace.

This is a generalisation of Kawamoto's approach.

<u>Totally & weakly ramified p-extension</u>

<u>Thm</u> L/K totally & weakly ramified p-extension $(p = \mathrm{char}\,\overline{K} > 0)$.
- (i) G elementary abelian p-group (standard)
- (ii) P_L^n free over $\mathcal{O}_K[G] \iff n \equiv 1 \mod |G|$ (known by Köck)
- (iii) Suppose $n \equiv 1 \mod |G|$.
  Then $\delta \in L$ free gen of P_L^n over $\mathcal{O}_K[G]$
  $\iff v_L(\delta) = n$.

(iii) Already shown by others (sometimes with restrictions) by Vostokov, Vinatier ($\neq$ Byott), Byott & Elder.

(In fact, works for perfect residue fields of +ve characteristic)

<u>Proof</u> Elementary
- Use Hilbert's formula to compute different of L/K.
- Obtain formula for $Tr_{L/K}(\mathfrak{B}_L^n)$.
- "Mod out" by $\mathfrak{B}_K$ $\rightsquigarrow$ work over $\bar{K}[G]$.
- Use (minor variant) of result of Childs
- Lift using Nakayama. $\square$

<u>Example</u> (if time)

$$\begin{tikzcd}
K(\zeta_{p^2}) \arrow[d, dash] \\
L \arrow[d, dash, "\text{weakly ram}"', "\text{degree } p"] \\
K \arrow[d, dash, "\text{unramified}"] \\
\mathbb{Q}_p
\end{tikzcd}$$

<u>Totally & weakly ramified extensions of arbitrary degree</u>

$$\begin{tikzcd}
L \arrow[d, dash, "I"] \\
K
\end{tikzcd}$$

$I = W \times C$ by Schur-Zassenhaus

$\uparrow$ wild inertia $= G_1$ = elementary abelian p-extension; C cyclic

$$\begin{tikzcd}
 & L \arrow[dl, dash, "W"'] \arrow[dr, dash, "C"] \arrow[dd, dash, "I"] & \\
E \arrow[dr, dash, "C"'] & & F \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

L/E, F/K totally weakly ram p-extensions
L/F, E/K totally and tamely ramified.

Define r by $|W| = p^r$, let $c = |C|$.
By Bézout $\exists a, b \in \mathbb{Z}$
s.t. $ap^r + bc = 1$.

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(Special case for simplicity.)

<u>Proposition</u> π_F any uniformizer $\quad \alpha := 1 + \pi_E + \pi_E^2 + \dots + \pi_E^{c-1}$.
Then $\pi_F^b \pi_E^a \cdot \alpha$ free gen of $\mathcal{P}_L$ over $\mathcal{O}_K[I]$.

<u>Proof</u>

(i) $v_L(\pi_F^b \pi_E^a) = 1$ so $\mathcal{P}_L = \mathcal{O}_E[W] \cdot (\pi_F^b \pi_E^a)$

(ii) $\pi_E^a \mathcal{O}_E = \mathcal{O}_K[C] \cdot (\pi_E^a \alpha)$

Do explicit calculation using semidirect product and that $\pi_F \in F = L^C$, $\pi_E, \alpha \in E = L^W$.

(Optional) $W = \{\tau_i\}$ $\quad C = \{\sigma_j\}$.
Starts like:

$$
\begin{aligned}
\mathcal{P}_L &= \mathcal{O}_E[W] \cdot (\pi_F^b \pi_E^a) && \text{(i)} \\
&= \bigoplus_i \tau_i (\pi_F^b \pi_E^a)\, \mathcal{O}_E \\
&= \bigoplus_i \tau_i (\pi_F^b) \cdot \pi_E^a \mathcal{O}_E && \text{since } \pi_E \in E = L^W \\
&= \quad (\text{use (ii)}) \dots \\
&= \dots \\
&= \bigoplus_i \bigoplus_j \tau_i \sigma_j (\pi_F^b \pi_E^a \cdot \alpha) \\
&= \mathcal{O}_K[I] \cdot (\pi_F^b \cdot \pi_E^a \cdot \alpha).
\end{aligned}
$$

<u>Rmk</u> If L/K abelian, totally & wildly ramified, not of p-power degree then L/K cannot be weakly ramified.

e.g.

not weakly ramified

$$
\begin{tikzcd}
\mathbb{Q}_p(\zeta_{p^2}) \arrow[d, dash, "p-1"] & \text{tame} \\
K \arrow[d, dash, "p"] & \\
\mathbb{Q}_p & \text{weakly ram}
\end{tikzcd}
$$

But $\mathbb{Q}_3(\zeta_3, \sqrt[3]{2}) / \mathbb{Q}_3$
Galois group S_3
Totally & weakly ramified.

(typo in paper)

Doubly split extensions

L/K finite Galois ext. of local fields.
$G = Gal(L/K)$ $I = G_0$ $W = G_1$
We say L/K is:
(i) split wrt inertia if $G = I \rtimes U$
for some (cyclic) U (so L/L^U unramified).
(ii) Split wrt wild inertia if $G = W \rtimes T$
for some T (so L/L^T tamely ramified).
(iii) doubly split if $\exists\, C \leq I$ and ~~[illegible]~~
$I = W \rtimes C$ $T = C \rtimes U$

so $G = W \rtimes T = W \rtimes (C \rtimes U) = (W \rtimes C) \rtimes U = I \rtimes U$

Rmk Automatic in totally ramified case by Schur Zassenhaus.

Glue generators together for doubly split extension.

Lemma L/K finite Galois ext of local fields.
Let K'/K be unique unramified extension of degree $[L:K]$.
Let $L' = LK'$. Then
(i) L'/K Galois
(ii) $Gal(L'/K')$ inertia subgroup
(iii) L'/K doubly split
(iv) if L/K weak ram then L'/K weak ram.

Proof Group theory. □

For general L/K, construct gen ε' for L'/K.
Then $\varepsilon := Tr_{L'/L}(\varepsilon)$ is gen for L/K.