Hopf orders in Hopf algebras with trivial Verschiebung

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Overview, Assumptions

Let R be a complete discrete valuation ring of (equal) characteristic p, $K = \text{Frac}\, R$.

We describe R-Hopf orders in a class of K-Hopf algebras H which are generated as K-algebras by their primitive elements $P(H)$.

[Some of this will work for $R = \mathbb{F}_q[t]$, $K = \mathbb{F}_q(t)$, q a power of p.]

These include orders in:

- $K[t]/(t^{p^n})$, the monogenic local-local Hopf algebra of rank p^n
- $(K\Gamma)^*$, Γ an elementary abelian p-group

Assumptions

- All group schemes are affine, flat, commutative, p-power rank.
- All Hopf algebras are abelian (commutative, cocommutative), free over its base ring, and of p-power rank.

Outline

Geometric Interpretation

For now, let R be any $\mathbb{F}_p$-algebra, $S = \mathrm{Spec}(R)$.

Let G be an S-group scheme. Then G is equipped with:

- The relative Frobenius morphism $F : G \to G^{(p)} := G \times_{S,\mathrm{Frob}} S$.
- The Verschiebung morphism $V : G^{(p)} \to G$, most easily defined as

$$V = (F_{G^\vee})^\vee$$

  where $^\vee$ indicates Cartier duality.

Note that $VF = p \cdot \mathrm{id}_G$ and $FV = p \cdot \mathrm{id}_{G^{(p)}}$.

Let $\mathbb{G}_{a,R}$ be the additive group scheme over R.

When R is understood, denote it $\mathbb{G}_a$.

Then $\mathrm{End}_{R\text{-gp}}(\mathbb{G}_a) \cong R[F]$, where $Fa = a^p F$ for all $a \in R$.

Given a finite group scheme G, define

$$\mathcal{D}^*(G) = \mathrm{Hom}_{R\text{-gp}}(G, \mathbb{G}_a).$$

The ring $R[F]$ acts on $\mathcal{D}^*(G)$ through its action on $\mathbb{G}_a$.

This gives a contravariant functor

$$\{\text{finite } R\text{-group schemes}\} \longrightarrow \{\text{finite } R[F]\text{-modules}\}$$

which is not an anti-equivalence.

$\mathcal{D}^*(G) = \mathrm{Hom}_{R\text{-gp}}(G, \mathbb{G}_a)$

However, the restricted functor

$$\left\{ \begin{array}{c} R\text{-group schemes} \\ \text{killed by } V \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{finite } R[F]\text{-modules,} \\ R\text{-free, killed by } V \end{array} \right\}$$

is an anti-equivalence; furthermore,

$$\mathrm{rk}(G) = p^{\mathrm{rk}_R(\mathcal{D}^*(G))},$$

and this is compatible with base change.

We will call finite, R-free $R[F]$-modules *Dieudonné modules*.

(It is the only type of Dieudonné module needed here.)

Q. Which finite group schemes are killed by V?

Finite subgroup schemes of $\mathbb{G}_a^n$ for some n.

(Short rationale: $\mathbb{G}_a = \ker V : W \to W$.)

Group schemes killed by p include:

- $\alpha_{p^n} = \ker F^n : \mathbb{G}_a \to \mathbb{G}_a$.
  (n^{th} Frobenius kernel of $\mathbb{G}_a$)
- $\mathbb{Z}/p\mathbb{Z} = \ker(F - \text{id}) : \mathbb{G}_a \to \mathbb{G}_a$.
  (constant group scheme)
- Finite products of the group schemes above.

This is not an exhaustive list.

Algebraic Interpretation

$\mathbb{G}_a = \mathrm{Spec}(R[t])$ with t primitive.

Let G be a group scheme, $G = \mathrm{Spec}(H)$.

Then

$$\mathcal{D}^*(G) = \mathrm{Hom}_{R\text{-gp}}(G, \mathbb{G}_a) \cong \mathrm{Hom}_{R\text{-Hopf alg}}(R[t], H).$$

Under this identification, $f \in \mathcal{D}^*(G)$ sends t to a primitive element in H, and f is completely determined by this image.

Thus, we define

$$\mathcal{D}_*(H) = P(H)$$

and obtain a categorical equivalence

$$\left\{ \begin{array}{c} \text{finite, flat, abelian} \\ \text{R-Hopf algebras} \\ \text{“killed by V"} \end{array} \right\} \longrightarrow \{\text{Dieudonné modules}\}\,.$$

The inverse

Let M be a finite $R[F]$-module, free over R. Let $\{e_1, e_2, \ldots, e_n\}$ be an R-basis for M.
Let $a_{i,j}$, $1 \leq i, j \leq n$ be given by

$$Fe_i = \sum_{j=1}^{n} a_{j,i} e_j.$$

Then $\mathcal{D}_*(H) = M$, where

$$H = R[t_1, \ldots, t_n]/(\{t_i^p - \sum_{j=1}^{n} a_{j,i} t_j\}), \{t_i\} \subset P(H)$$

By writing $M = R^n$ and using e_i as a standard basis vector, we have

$$Fe_i = Ae_i$$

where $A = (a_{i,j}) \in M_n(R)$.

Some Examples

Throughout, we also use F to denote the Frobenius morphism on Hopf algebras.

In each example, the explicit algebra generators are primitive.

Example

$G = \alpha_p^n$, $H = R[t_1, \ldots, t_n]/(t_1^p, \ldots, t_n^p)$. $P(H) = \mathrm{Span}_R\{t_1, \ldots, t_n\}$.
$F(t_i^p) = 0, 1 \leq i \leq n$.
So $\mathcal{D}_*(H)$ is R-free on $e_1, \ldots, e_n$ with

$$Fe_i = 0.$$

In this case, $A = 0$ ($Fe_i = Ae_i = 0$).

Example

$G = \alpha_{p^n}$, $H = R[t]/(t^{p^n}) = R[t_1, \ldots, t_n]/(t_1^p, t_2^p - t_1, \ldots, t_n^p - t_{n-1})$
$P(H) = \mathrm{Span}_R\{t, t^p, \ldots, t^{p^{n-1}}\}$.
$F(t^{p^i}) = t^{p^{i+1}}, 0 \leq i \leq n-1$.
So $\mathcal{D}_*(H)$ is R-free on $e_1, \ldots, e_n$ with

$$Fe_i = \begin{cases} e_{i-1} & i > 1 \\ 0 & i = 1 \end{cases}.$$

In this case,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Example

$G = (\mathbb{Z}/p\mathbb{Z})^n$, $H = \left(RC_p^n\right)^* = R[t_1, \ldots, t_n]/(t_1^p - t_1, \ldots, t_n^p - t_n)$
$P(H) = \mathrm{Span}_R\{t_1, \ldots, t_n\}$.
$F(t_i) = t_i^p = t_i, 1 \leq i \leq n$.
So $\mathcal{D}_*(H)$ is R-free on $e, \ldots, e_n$ with $Fe_i = e_i$ for all i.
Clearly, $A = I$.

An example in the other direction

Example

Let A be the cyclic permutation matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \mathcal{D}(H) = M_A.$$

Then H is generated by primitive elements $t_1, \ldots, t_n$ with

$$t_i^p = \begin{cases} t_{i-1} & i > 0 \\ t_n & i = 0 \end{cases}$$

If we set $t = t_n$ then

$$H = R[t]/(t^{p^n} - t),$$

Outline

Maps

Let $A, B \in M_n(R)$.

Let M_A, M_B be free R-modules of rank n which are also $R[F]$-modules via

$$Fe_i = Ae_i \text{ and } Fe_i = Be_i$$

respectively.

A morphism of Dieudonné modules is an R-linear map $M_A \to M_B$ which respects the actions of F.

Let $\Theta \in M_n(R)$ represent (and be) an R-linear map $M_A \to M_B$.

Let $n = 2$ and write

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \Theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}.$$

Then:

$$\begin{aligned} F(\Theta(e_1)) &= F(\theta_1 e_1 + \theta_3 e_2) \\ &= \theta_1^p(b_1 e_1 + b_3 e_2) + \theta_3^p(b_2 e_1 + b_4 e_2) \\ &= (\theta_1^p b_1 + \theta_3^p b_2)e_1 + (\theta_1^p b_3 + \theta_3^p b_4)e_2 \\ \Theta(Fe_1) &= \Theta(a_1 e_1 + a_3 e_2) \\ &= a_1(\theta_1 e_1 + \theta_3 e_2) + a_3(\theta_2 e_1 + \theta_4 e_2) \\ &= (a_1\theta_1 + a_3\theta_2)e_1 + (a_1\theta_3 + a_3\theta_4)e_2. \end{aligned}$$

Repeat for $F(\Theta(e_2)) = \Theta(Fe_2)$. We get:

$$\begin{aligned}
\theta_1 a_1 + \theta_2 a_3 &= b_1\theta_1^p + b_2\theta_3^p \\
\theta_3 a_1 + \theta_4 a_3 &= b_3\theta_1^p + b_4\theta_3^p \\
\theta_1 a_2 + \theta_2 a_4 &= b_1\theta_2^p + b_2\theta_4^p \\
\theta_3 a_2 + \theta_4 a_4 &= b_3\theta_2^p + b_4\theta_4^p.
\end{aligned}$$

In other words,

$$\Theta A = B\Theta^{(p)}$$

where $\Theta^{(p)} = (\theta_i^p)$ for all i.

Furthermore, Θ is an isomorphism if and only if $\Theta \in M_2(R)^\times$.

This generalizes to any n.

Choosing $A \in M_n(R)$ gives an R-Hopf algebra, say H_A.
But. Different choices of A can produce the "same" Hopf algebra.

Example

Pick $r \in R, r \notin \mathbb{F}_p$, and let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ r^p - r & 1 \end{pmatrix}.$$

Then

$$H_A = R[t_1, t_2]/(t_1^p - t_1, t_2^p - t_2)$$
$$H_B = R[u_1, u_2]/(u_1^p - u_1 - (r^p - r)u_2, u_2^p - u_2)$$

Since

$$(t_1 + rt_2)^p = t_1 + r^p t_2 = t_1 + rt_2 + r^p t_2 - rt_2 = (t_1 + rt_2) + (r^p - r)t_2$$

if we let $u_1 = t_1 + rt_2$, $u_2 = t_2$, then $H_A = H_B$.

Outline

From now on, $R = \mathbb{F}_q[[T]]$, $K = \mathbb{F}_q((T))$.

Let v_K be the valuation on K with $v_K(T) = 1$.

We have R-Dieudonné modules and K-Dieudonné modules, compatible with base change.

Pick $A, B \in M_n(R)$ and construct R-Dieudonné modules M_A, M_B.

Write $M_A = \mathcal{D}_*(H_A)$ and $M_B = \mathcal{D}_*(H_B)$.

Then $\mathcal{D}_*(KH_A)$ is a Dieudonné module over K and

$$\mathcal{D}_*(KH_A) \cong \mathcal{D}_*(H_A) \otimes_R K.$$

Similarly,

$$\mathcal{D}_*(KH_B) \cong \mathcal{D}_*(H_B) \otimes_R K.$$

Now $KH_A \cong KH_B$ if and only if there is a $\Theta \in \mathrm{GL}_n(K)$ which, viewed as a K-linear isomorphism

$$\mathcal{D}_*(KH_A) \to \mathcal{D}_*(KH_B),$$

respects the F-actions on the K-Dieudonné modules.

Thus, H_A and H_B are Hopf orders in the same K-Hopf algebra iff

$$\Theta A = B\Theta^{(p)} \text{ for some } \Theta \in \mathrm{GL}_n(K).$$

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Write $A = (a_{i,j}), B = (b_{i,j}), \Theta = (\theta_{i,j})$. Then

- H_A is viewed as an R-Hopf algebra using A, i.e.

$$H_A = R[u_1, \dots u_n]/(\{u_i^p - \sum a_{j,i}u_j\}).$$

- H_B is viewed as an R-Hopf algebra using B, i.e.

$$H_B = R[t_1, \dots t_n]/(\{t_i^p - \sum b_{j,i}t_j\}).$$

- KH_B is viewed as a K-Hopf algebra in the obvious way.
- H_B is viewed as an order in KH_B in the obvious way.
- H_A is viewed as an order in KH_B through Θ, i.e.

$$H_A = R\left[\left\{\sum_{j=1}^{n} \theta_{j,i}t_j : 1 \le i \le n\right\}\right] \subset KH_B$$

(apologies for the abuses of language)

Outline

What are the rank Hopf algebras (killed by V) over K?

They correspond to rank 1 Dieudonné modules over K.

Fix $a \in K$, and let $M = Ke$ with $Fe = ae$.

The corresponding Hopf algebra is $H_a := K[t]/(t^p - at)$.

Furthermore, $H_a \cong H_b$ if and only if there is a $\theta \in K^\times$ with $\theta a = b\theta^p$.

Case $b = 0$**.** This is the algebra which represents α_p.

Case $b \neq 0$**.** Let $\theta = b^{1/(1-p)} \in K^{\text{sep}}$.

Then

$$H_b \otimes K^{\text{sep}} \cong H_1 \otimes K^{\text{sep}} = (K^{\text{sep}} C_p)^*,$$

hence H_b is a form of KC_p^*.

(This is a well-known result in descent theory.)

Case $b = 1$: orders in KC_p^*

Strategy. Pick $\theta \in K$ such that $\theta^{p-1} \in R$ (so $\theta \in R$), and let $a = \theta^{p-1}$.
Then $M = Re$, $Fe = ae$ is an R-Dieudonné module, whose Hopf algebra H_θ is generically isomorphic to KC_p^*.
The R-Hopf algebra is

$$H_\theta \cong R[u]/(u^p - au).$$

We can view it as a Hopf order by identifying u with θt and hence

$$H_\theta = R[\theta t] \subset K[t]/(t^p - t).$$

Check:

$$u^p = (\theta t)^p = \theta^p t^p = \theta^{p-1} \theta t = au.$$

Note $H_{\theta_1} = H_{\theta_2}$ iff $v_K(\theta_1) = v_K(\theta_2)$, so the Hopf orders are:

$$H_i = R[T^i t], i \geq 0.$$

Case $b \neq 0$

Since $K[t]/(t^p - bt) \cong K[t]/(t^p - T^{p-1}bt)$ by the map $t \mapsto T^{-1}t$, we may assume

$$0 \leq v_K(b) < p - 1.$$

Pick $\theta \in K^\times$ and let $a = b\theta^{p-1}$.
Provided $a \in R$ (which holds iff $\theta \in R$) we have

$$H_\theta = R[\theta t] \subset K[t]/(t^p - bt).$$

As before, if $u = \theta t$ then

$$u^p = (\theta t)^p = \theta^p t^p = \theta^{p-1} b\theta t = au$$

and so $H_\theta = R[u]/(u^p - au)$.
Again, H_θ depends only on $v_K(\theta)$, so a complete list is

$$H_i = R\left[T^i t\right], i \geq 0.$$

Case $b = 0$

Clearly, $\theta a = b\theta^p$ can occur only if $a = 0$.

So, any two R-Hopf orders in $K[t]/(t^p)$ are isomorphic.

However. They are not necessarily the same Hopf order. It depends on the chosen embedding $\Theta = (\theta) \in GL_1(K)$.

Let $\theta \in K^\times$.

Then $R[\theta t]$ is a Hopf order in $K[t]/(t^p)$.

As $R[\theta t] = R[(r\theta)t], r \in R^\times$, the complete list is

$$H_i = R[T^i t], i \in \mathbb{Z}.$$

Outline

The new definition of "all"

The R-Hopf orders we find are subgroups of $\mathbb{G}^n_{a,R}$.

If there are nontrivial models of $\mathbb{G}^n_{a,K}$ then it we may miss some Hopf orders.

i.e., If $\mathcal{G} = \mathrm{Spec}(A)$, $\mathcal{G} \ncong \mathbb{G}_{a,R}$ with

$$A \otimes_R K \cong R[x_1, \ldots, x_n], \Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i,$$

and $G{=}\mathrm{Spec}(H)$ is a subgroup of $\mathbb{G}^n_{a,K}$, and $G_0{=}\mathrm{Spec}(H_0)$ is a subgroup of $\mathcal{G}$ such that H_0 is a Hopf order in H, our procedure will not detect it.

This is not an issue when $n = 1$ since $\mathbb{G}_{a,K}$ has no nontrivial models.

Overview for all n

- Pick a K-Hopf algebra H, and find the $B \in M_n(K)$ which is used in the construction of its K-Dieudonné module.
- Find $A \in M_n(R)$ such that $\Theta A = B\Theta^{(p)}$ for some $\Theta \in \mathrm{GL}_n(K)$. (One such example: $A = B$, $\Theta = I$.)
- Construct the R-Dieudonné module corresponding to A.
- Construct the R-Hopf algebra H_A corresponding to this Dieudonné module.
- The algebra relations on H_A are given by the matrix A.
- H_A can be viewed as a Hopf order in H using Θ.
- $H_{A_1} = H_{A_2}$ if and only if $\Theta^{-1}\Theta'$ is an invertible matrix in R, where

$$\Theta A_1 = B\Theta^{(p)} \text{ and } \Theta' A_2 = B(\Theta')^{(p)}.$$

Alternatively, $H_{A_1} = H_{A_2}$ if and only if $\Theta' = \Theta U$ for some $U \in M_n(R)^{\times}$.

$\Theta A = B\Theta^{(p)}$

Note. $M_n(R)^\times$ are the matrices that invert in R, not invertible matrices with entries in R.

$$M_n(R)^\times \subsetneq M_2(R) \cap GL_2(K).$$

One strategy. Given B, set $A = \Theta^{-1}B\Theta^{(p)}$.

This will generate a Hopf order iff A has coefficients in R.

But, we can replace Θ with ΘU for $U \in M_2(R)^\times$.

Notice that

$$(\Theta U)^{-1}B(\Theta U)^{(p)} = U^{-1}\left(\Theta^{-1}B\Theta^{(p)}\right)U^{(p)},$$

and so $(\Theta U)^{-1}B(\Theta U)^{(p)} \in M_n(R)$ iff $\Theta^{-1}B\Theta^{(p)} \in M_n(R)$.

n is now 2

Write $\Theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}$.

If $v_K(\theta_2) < v_K(\theta_1)$ then replace Θ with

$$\Theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so $v_K(\theta_2) \geq v_K(\theta_1)$.

Then replace this (possibly new) Θ with

$$\Theta \begin{pmatrix} 1 & -\theta_2/\theta_1 \\ 0 & 1 \end{pmatrix}$$

so we may assume $\theta_2 = 0$.

Now Θ is lower triangular.

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We can replace Θ with

$$\Theta \begin{pmatrix} \theta_1^{-1} T^{v_K(\theta_1)} & 0 \\ 0 & \theta_4^{-1} T^{v_K(\theta_4)} \end{pmatrix}$$

to make θ_1, θ_4 powers of T.

Finally, if $v_K(\theta_4) \leq v_K(\theta_3)$ then we can replace Θ with

$$\Theta \begin{pmatrix} 1 & 0 \\ -\theta_3/\theta_4 & 1 \end{pmatrix}$$

to make $\theta_3 = 0$.

So, we have two cases:

1. Θ is diagonal.
2. Θ is lower triangular with $v_K(\theta_3) < v_K(\theta_4)$.

Set $\theta_1 = T^i, \theta_2 = 0, \theta_3 = \theta, \theta_4 = T^j$

The Hopf orders will be of the form

$$H_{i,j,\theta} = R\left[T^i t_1 + \theta t_2, T^j t_2\right]$$

with $\theta = 0$ or $v_K(\theta) < j$.

But, not every expression of this form is a Hopf order.

Example

$H = K[t_1, t_2]/(t_1^p - t_1, t_2^p - t_2)$
Then $H_{-1,0,0} = R\left[T^{-1} t_1, t_2\right]$ is not a Hopf order because it is not a finitely generated R-module, e.g.:

$$(T^{-1}t_1)^p = T^{-p}t_1 \notin \text{Span}_R\{(T^{-1}t_1)^i t_2^j : 0 \leq i, j \leq p-1\}.$$

$H_{i,j,\theta} = R\left[T^i t_1 + \theta t_2, T^j t_2\right], v_K(\theta) < j$

In the case $\theta = 0$ we get

$$H_{i,j,0} = R\left[T^i t_1, T^j t_2\right],$$

Creating *Larson-like* orders.

Note that we can (and often do) have $i, j > 0$, in contrast to the “real" Larson orders.

$H_{i,j,\theta} = R\left[T^i t_1 + \theta t_2, T^j t_2\right], v_K(\theta) < j$

Q. When is $H_{i,j,\theta} = H_{i',j',\theta'}$?

Precisely when there is a $U \in M_n(R)^\times$ such that

$$\begin{pmatrix} T^i & 0 \\ \theta & T^j \end{pmatrix} = \begin{pmatrix} T^{i'} & 0 \\ \theta' & T^j \end{pmatrix} U.$$

Such a U exists if and only if

$$\begin{aligned} i &= i' \\ j &= j' \\ v_K(\theta - \theta') &\geq j. \end{aligned}$$

Note that this includes the case $\theta = 0$ since for $\theta' \neq 0$, $v_K(\theta') < j$.

An example: $H = K[t_1, t_2], (t_1^p, t_2^p)$

$$A = \Theta^{-1} B \Theta^{(p)}$$

Here $B = 0$, so we must have $A = 0$.
Then any $\Theta \in \mathrm{GL}_2(K)$ gives a Hopf order.
In this case,

$$H_{i,j,0} = R\left[T^i t_1, T^j t_2\right]$$

$$H_{i,j,\theta} = R\left[T^i t_1 + \theta t_2, T^j t_2\right], i, j \in \mathbb{Z}, v_K(\theta) < j$$

are all of the Hopf orders, and $H_{i,j,\theta} = H_{i,j,\theta'}$ iff $v_K(\theta - \theta') \geq j$.
Writing $\theta = T^k u$, $v_K(u) = 0$ gives a parameterization of all of the non-Larson-like Hopf orders:

$$\{(i, j, k, u) : i, j, k \in \mathbb{Z}, k < j, 0 \neq u \in R/T^{j-k}R\}$$

Another example: $H = K[t]/(t^{p^2}) = K[t_1, t_2]/(t_1^p, t_2^p - t_1)$

In this case, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let $\Theta = \begin{pmatrix} T^i & 0 \\ \theta & T^j \end{pmatrix}$, $\theta = 0$ or $v_K(\theta) < j$.

$$A = \Theta^{-1} B \Theta^{(p)} = \begin{pmatrix} \theta^p T^{-i} & T^{pj-i} \\ -\theta^{p+1} T^{-(i+j)} & \theta T^{(p-1)j-i} \end{pmatrix}.$$

If $\theta = 0$, then $pj \geq i$, so $A = \begin{pmatrix} 0 & T^{pj-i} \\ 0 & 0 \end{pmatrix}$, which gives the Larson-like Hopf order

$$R[T^i t_1, T^j t_2] = R[T^i t^p, T^j t],$$

which is monogenic if and only if $pj = i$.

$H = K[t]/(t^{p^2}) = K[t_1, t_2]/(t_1^p, t_2^p - t_1)$ (still)

Case $v_K(\theta) < j$:

$$A = \begin{pmatrix} \theta^p T^{-i} & T^{pj-i} \\ -\theta^{p+1} T^{-(i+j)} & \theta T^{(p-1)j-i} \end{pmatrix}$$

To give a Hopf order, we require $pj > i$ and

$$v_K(\theta) \geq \min\{i/p, (i+j)/(p+1), i-(p-1)j\},$$

giving

$$R = \left[T^i t_1 + \theta t_2, T^j t_2\right] = R\left[T^i t^p + \theta t, T^j t\right].$$

Note. If $pj = i$ then

$$i/p = j \leq v_K(\theta) < j$$

which can't happen.

Yet another example: $H = K[t_1, t_2]/(t_1^p - t_1, t_2^p - t_2)$

This is $K(C_p \times C_p)^*$.
Here, $B = I$, so pick Θ and set $A = \Theta^{-1}\Theta^{(p)}$:

$$A = \frac{1}{T^{i+j}} \begin{pmatrix} T^{pi+j} & 0 \\ T^i\theta^p - T^{pi}\theta & T^{i+pj} \end{pmatrix} = \begin{pmatrix} T^{(p-1)i} & 0 \\ T^{-j}\theta^p - T^{(p-1)i-j}\theta & T^{(p-1)j} \end{pmatrix}.$$

The Larson-likes are easy to find.

$$\Theta = \begin{pmatrix} T^i & 0 \\ 0 & T^j \end{pmatrix} \Rightarrow A = \begin{pmatrix} T^{(p-1)i} & 0 \\ 0 & T^{(p-1)j} \end{pmatrix}.$$

Clearly, $A \in M_2(R)$ if and only if $i, j \geq 0$.
Thus, the Larson-like Hopf orders we get are

$$H_{i,j} = R\left[T^i t_1, T^j t_2\right], i, j \geq 0.$$

$H = K[t_1, t_2]/(t_1^p - t_1, t_2^p - t_2)$, Θ not diagonal

$$A = \begin{pmatrix} T^{(p-1)i} & 0 \\ T^{-j}\theta^p - T^{(p-1)i-j}\theta & T^{(p-1)j} \end{pmatrix}.$$

Again, $i, j \geq 0$. Let $k = v_K(\theta)$. For $A \in M_2(R)$ we also need

$$v_K(\theta^{p-1} - T^{(p-1)i}) \geq j - k \text{ (note } j - k > 0\text{)},$$

and this suffices.

Thus,

$$H_{i,j,\theta} = R\left[T^i t_1 + \theta t_2, T^j t_2\right]$$

where $\theta^{p-1} \equiv T^{(p-1)i} \bmod T^{j-k}R$, i.e.,

$$\theta \equiv zT^i \bmod T^{\lfloor (j-k)/(p-1) \rfloor}R, z \in \mathbb{F}_p^\times.$$

This example was introduced earlier, now specialized to $n = 2$.

Here, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$A = \begin{pmatrix} \theta^p T^{-i} & T^{pj-i} \\ T^{pi-j} - \theta^{p+1} T^{-(i+j)} & -\theta T^{(p-1)j-i} \end{pmatrix}.$$

The Larson-like orders are

$$R\left[T^i t_1, T^j t_2\right] = R\left[T^i t^p, T^j t\right], i \le pj \le p^2 i.$$

Note. We have $i, j \ge 0$, and the order is monogenic iff $pj = i$.

$$A = \begin{pmatrix} \theta^p T^{-i} & T^{pj-i} \\ T^{pi-j} - \theta^{p+1} T^{-(i+j)} & -\theta T^{(p-1)j-i} \end{pmatrix}.$$

There are numerous non-Larson-like orders, but they remain somewhat cumbersome to describe. Along with $pj \geq i$ we need

$$\begin{aligned} v_K(\theta) &\geq i/p \\ v_K(\theta) &\geq i - j(p-1) \\ v_K(T^{(p+1)i-2j} - \theta^{p+1}) &\geq i + j \end{aligned}$$

From this, we know, e.g.,

$$\begin{aligned} &j - (pj - i) \leq v_K(\theta) < j \\ &i/p \leq v_K(\theta) < j, \end{aligned}$$

but these are not sufficient inequalities.

Outline

Possible Directions

1. Extension of these examples to arbitrary n.

The Larson-likes seem easy for all.

The non-Larson-likes seem doable for

- $K[t]/(t^{p^n})$
- $K[t_1, \ldots, t_n]/(t_1^p, t_2^p, \ldots, t_n^p)$,

more complicated for $(RC_p^n)^*$.

Need a simple representation for Θ. (Lower triangular? Powers of T on diagonal? Highest valuation on diagonal?)

Possible Directions

2. Make concrete connections to works with similar results from characteristic zero.

- Construction of Hopf orders in KC_{p^n} and KC_p^n using polynomial formal groups. [Childs et al]
- Breuil-Kisin module constructions corresponding to Hopf orders in KC_p^n [K.]

Both of these use a matrix Θ, and the location of the entries of $\Theta^{-1}\Theta^{(p)}$ is important.

Possible Directions

3. Extend to cases where V does not act trivially.

Possible tools:

- Breuil-Kisin modules
- Dieudonné displays, frames, etc.
- 1993 work of de Jong, in which the Dieudonné correspondence here can be found, treats more general cases to some degree.

There is also an equivalence between group schemes $G = \mathrm{Spec}(H)$ killed by F and $R[V]$-modules given by

$$H \mapsto H^+ / \left(H^+\right)^2 .$$

This correspondence may be used to find Hopf orders in, for example, elementary abelian group rings.

Thank you.