Hopf-Galois Theory and Galois Module Structure
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Induced Hopf Galois structures

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Let K/k be a separable field extension of degree n, $\widetilde{K}$ its Galois closure, $G = \mathrm{Gal}(\widetilde{K}/k)$, $G' = \mathrm{Gal}(\widetilde{K}/K)$. A Hopf Galois structure on K/k may be given, equivalently, by

- a finite cocommutative k-Hopf algebra $\mathcal{H}$ and a Hopf action of $\mathcal{H}$ on K, i.e a k-linear map $\mu : \mathcal{H} \to \mathrm{End}_k(K)$ inducing a bijection $K \otimes_k \mathcal{H} \to \mathrm{End}_k(K)$. (Chase-Sweedler)
- a regular subgroup N of S_n normalized by $\lambda(G)$, where $\lambda : G \to S_n$ is the morphism given by the action of G on the left cosets G/G'. (Greither-Pareigis)

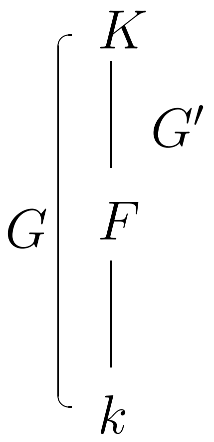
  If $N \subset \lambda(G)$, equivalently N is a normal complement of G' in G, K/k is called almost classically Galois.
- a group monomorphism $\varphi : G \to \mathrm{Hol}(N)$ such that $\varphi(G')$ is the stabilizer of 1_N, where $\mathrm{Hol}(N) = N \rtimes \mathrm{Aut}\, N \hookrightarrow \mathrm{Sym}(N)$ is defined by sending $n \in N$ to left translation by n and $\sigma \in \mathrm{Aut}\, N$ to itself. (Childs, Byott)

$N \subset S_n$ regular, normalized by $\lambda(G) \leftrightarrow \mathcal{H} = \widetilde{K}[N]^G$

$\{$ Hopf subalgebras of $\mathcal{H}\} \leftrightarrow \{G$-stable subgroups of $N\}$

For N' a G-stable subgroup of N, $K^{N'} := K^{\mathcal{H}'}$ for $\mathcal{H}'$ the Hopf subalgebra of $\mathcal{H}$ corresponding to N'.

Theorem 1.



K/k finite Galois

$G = \mathrm{Gal}(K/k)$

$G' = \mathrm{Gal}(K/F)$

$G = G' \ltimes H$

$r = [K : F], t = [F : k], n = [K : k]$

Assume that

- N_1 gives F/k a Hopf Galois structure and
- N_2 gives K/F a Hopf Galois structure.

Then $N_1 \times N_2$ gives K/k a Hopf Galois structure.

Proof.
N_1 gives F/k a Hopf Galois structure $\Leftrightarrow \exists \varphi_1 : G \to \mathrm{Hol}(N_1)$, with kernel $\mathrm{Gal}(K/\widetilde{F})$, such that $\varphi_1(G') = Stab(1_{N_1})$.
N_2 gives K/F a Hopf Galois structure $\Leftrightarrow \exists \varphi_2 : G' \hookrightarrow \mathrm{Hol}(N_2)$ such that $\varphi_2(1_{G'}) = Stab(1_{N_2})$.
If $g, g' \in G, g = xy, g' = x'y'$ with $x, x' \in H, y, y' \in G'$, since $H \lhd G$, we have $gg' = (xx'')yy'$, for some $x'' \in H$. Hence, the map

$$\begin{array}{rccl} \overline{\varphi} : & G & \to & \mathrm{Hol}(N_1) \times \mathrm{Hol}(N_2) \\ & g = xy & \mapsto & (\varphi_1(g), \varphi_2(y)) \end{array}$$

is a group monomorphism. We define now

$$\begin{array}{rcl} \iota : \mathrm{Hol}(N_1) \times \mathrm{Hol}(N_2) & \hookrightarrow & \mathrm{Hol}(N_1 \times N_2) \\ ((n_1, \sigma_1), (n_2, \sigma_2)) & \mapsto & ((n_1, n_2), \sigma) \end{array}$$

where $\sigma(n_1, n_2) := (\sigma_1(n_1), \sigma_2(n_2))$, and consider

$$\varphi : G \xrightarrow{\overline{\varphi}} \mathrm{Hol}(N_1) \times \mathrm{Hol}(N_2) \overset{\iota}{\hookrightarrow} \mathrm{Hol}(N_1 \times N_2).$$

We check $\varphi(1_G) = Stab(1_{N_1 \times N_2})$: for $g = xy \in G, x \in H, y \in G'$, $\varphi(g)(1_{N_1 \times N_2}) = 1_{N_1 \times N_2} \Leftrightarrow \varphi_1(g)(1_{N_1}) = 1_{N_1}$ and $\varphi_2(y)(1_{N_2}) = 1_{N_2} \Leftrightarrow g \in G'$ and $y = 1_{G'} \Leftrightarrow g = 1_G$.

A Hopf Galois structure on a Galois extension K/k with Galois group G will be called

induced if it is obtained as in Theorem 1 for some field F with $k \subsetneq F \subsetneq K$ and given Hopf Galois structures on F/k and K/F;

split if the corresponding regular subgroup of $Sym(G)$ is the direct product of two nontrivial subgroups.

Corollary. A Galois extension K/k with Galois group $G = H \rtimes G'$ has at least one split Hopf Galois structure of type $H \times G'$.

Proof. Let $F = K^{G'}$ and let $\widetilde{F}$ be the normal closure of F in K. Then K/F is Galois with group G' and F/k is almost classically Galois of type H since H is a normal complement of $\mathrm{Gal}(\widetilde{F}/F)$ in $\mathrm{Gal}(\widetilde{F}/k)$. These two Hopf Galois structures induce a Hopf Galois structure on K/k of type $H \times G'$.

A Galois extension with Galois group G has an induced Hopf Galois structure of type N in each of the following cases.

G	N
$S_3 = C_3 \rtimes C_2$	$C_6 = C_3 \times C_2$
$D_{2n} = C_n \rtimes C_2$	$C_n \times C_2$
$S_n = A_n \rtimes C_2$	$A_n \times C_2$
$A_4 = V_4 \rtimes C_3$	$V_4 \times C_3$
Frobenius group $G = H \rtimes G'$	$H \times G'$
$\mathrm{Hol}(M) = M \rtimes \mathrm{Aut}(M)$	$M \times \mathrm{Aut}(M)$

A Frobenius group G is a transitive permutation group of some finite set X, such that every $g \in G \setminus \{1\}$ fixes at most one point of X and some $g \in G \setminus \{1\}$ fixes a point of X. We have $G = H \rtimes G'$, where H is the Frobenius kernel, i.e. the subgroup of G whose nontrivial elements fix no point of X, and G' is a Frobenius complement, i.e. the stabilizer of one point of X.
A semi-direct product $G = H \rtimes G'$ is a Frobenius group iff $C_G(h) \subset H$ for all $h \in H \setminus \{1\}$, and $C_G(g') \subset G'$ for all $g' \in G' \setminus \{1\}$.

Split non-induced Hopf Galois structures

1. The quaternion group

$$H_8 = \langle i, j | i^4 = 1, i^2 = j^2, ij = ji^3 \rangle = \{1, i, i^2, i^3, j, ij, i^2j, i^3j\}$$

is not a semi-direct product of two subgroups. The action of H_8 on itself by left translation induces

$$\begin{array}{rcl} \lambda : H_8 & \to & \mathrm{Sym}(H_8) \\ i & \mapsto & (1, i, i^2, i^3)(j, ij, i^2j, i^3j) \\ j & \mapsto & (1, j, i^2, i^2j)(i, i^3j, i^3, ij) \end{array}$$

Then, $\lambda(H_8)$ normalizes

$$N = \langle (1, i^2)(i, i^3)(j, i^2j)(ij, i^3j), (1, i^3)(i, i^2)(i, ij)(i^2j, i^3j), (1, i^3j)(i, j)(i^2, ij)(i^3, i^2j) \rangle$$

which is a regular subgroup of $\mathrm{Sym}(H_8)$ isomorphic to $C_2 \times C_2 \times C_2$. Hence a Galois extension with Galois group H_8 has a split Hopf Galois structure of type $C_2 \times C_2 \times C_2$.

2. In the case $G = H \times G'$, i.e. F/k Galois, the Galois structures of K/F and F/k induce the Galois structure on K/k:

$$G \to G/G' = H \xrightarrow{\rho} \mathrm{Hol}(H) \text{ and } G' \xrightarrow{\rho} \mathrm{Hol}(G') \text{ give } G \xrightarrow{\rho} \mathrm{Hol}(G).$$

Let us consider a Galois extension K/k with Galois group $G \simeq C_p \times C_p$ (with p prime). There are p^2 different Hopf Galois structures for K/k (Byott,1996).

Case $p = 2$**:** There is only one structure of type $C_2 \times C_2$, which is the classical one. The remaining 3 are of cyclic type. The extension K/k has 3 different quadratic subextensions but all of them give rise to the same Hopf Galois structure, corresponding to $N = V_4 \subset S_4$.

Case $p > 2$**:** $\mathrm{Hol}(C_{p^2})$ has no transitive subgroup isomorphic to $C_p \times C_p$. All p^2 Hopf Galois structures are split: $N \simeq C_p \times C_p$. Only the classical structure is induced. The extension K/k has $p+1$ different subextensions of degree p but all of them give rise to the classical structure.

We obtain then that a split Hopf Galois structure on a Galois extension K/k may be induced by Hopf Galois structures on K/F and F/k, for different intermediate fields F.

Given a Galois extension K/k of degree n with Galois group G and a regular subgroup $N = N_1 \times N_2$ of S_n giving K/k a split Hopf Galois structure, *under which conditions is this Hopf Galois structure induced?*

Theorem 1 gives that the following conditions are necessary.

1) N_1 and N_2 are G-stable,

2) If $F = K^{N_2}$ and $G' = \mathrm{Gal}(K/F)$, then G' has a normal complement in G.

Theorem 2. Let K/k be a finite Galois field extension, $n = [K : k]$, $G = \mathrm{Gal}(K/k)$. Let K/k be given a split Hopf Galois structure by a regular subgroup N of S_n such that $N = N_1 \times N_2$ with N_1 and N_2 G-stable subgroups of N. Let $F = K^{N_2}$ be the subfield of K fixed by N_2 and let us assume that $G' = \mathrm{Gal}(K/F)$ has a normal complement in G.
Then K/F is Hopf Galois with group N_2 and F/k is Hopf Galois with group N_1. Moreover the Hopf Galois structure of K/k given by N is induced by the Hopf Galois structures given by N_1 and N_2.

Proof. Since K/k is Hopf Galois with group N, we have a monomorphism

$$\begin{array}{rcl} \varphi: G & \to & \mathrm{Hol}(N) = N \rtimes \mathrm{Aut}\, N \\ g & \mapsto & \varphi(g) = (n(g), \sigma(g)) \end{array}$$

such that $\varphi(1_G)$ is the stabilizer of 1_N.
Let us see $\varphi(G) \subset \iota(\mathrm{Hol}(N_1) \times \mathrm{Hol}(N_2))$, for

$$\begin{array}{rcl} \iota: \mathrm{Hol}(N_1) \times \mathrm{Hol}(N_2) & \hookrightarrow & \mathrm{Hol}(N_1 \times N_2) \\ ((n_1, \sigma_1), (n_2, \sigma_2)) & \mapsto & ((n_1, n_2), \sigma). \end{array}$$

For $i = 1, 2$, N_i G-stable and $N_i \lhd N \Rightarrow$

$$\text{for } \; n_i \in N_i, g \in G,\ n(g)\sigma(g)(n_i)n(g)^{-1} \in N_i \Rightarrow \sigma(g)(n_i) \in N_i.$$

We obtain then morphisms

$$\begin{array}{rcl} \varphi_1: G & \to & \mathrm{Hol}(N_1) \\ g & \mapsto & (\pi_1(n(g)), \sigma(g)_{|N_1}) \end{array} \qquad \begin{array}{rcl} \varphi_2: G' & \to & \mathrm{Hol}(N_2) \\ g & \mapsto & (\pi_2(n(g)), \sigma(g)_{|N_2}) \end{array}$$

Since $F = K^{N_2}$ and $G' = \mathrm{Gal}(K/F)$, we have for $g \in G$, $g \in G' \Leftrightarrow \varphi(g)(1_N) \in N_2$.
Hence $\varphi_1(G') = Stab(1_{N_1})$.
Now for $y \in G', \varphi_2(y)(1_{N_2}) = 1_{N_2} \Rightarrow \varphi_2(y)(1_N) \in N_1$. But we had $\varphi(y)(1_N) \in N_2$, hence $\varphi(y)(1_N) = 1_N$, which implies $y = 1_G$, so $\varphi_2(1_{G'}) = Stab(1_{N_2})$.

Counting Hopf Galois structures

1. The alternating group A_4

K/k Galois with group A_4 has only two types of Hopf Galois structures: A_4 and $V_4 \times C_3$.

$$e(A_4, A_4) = 10 \text{ (Carnahan-Childs, 1999).}$$

Let us determine the number of induced Hopf Galois structures of type $V_4 \times C_3$.
We have a unique choice for the nontrivial normal subgroup H, the Klein subgroup $V_4 = \{id, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. It has four different complements in G

$$G'_1 = \langle (2,3,4) \rangle, \; G'_2 = \langle (1,3,4) \rangle, \; G'_3 = \langle (1,2,4) \rangle, \; G'_4 = \langle (1,2,3) \rangle.$$

For a fixed G', $F = K^{G'}/k$ is a quartic extension with Galois closure K and has a unique Hopf Galois structure of type V_4 given by $\varphi_1 : A_4 \hookrightarrow \mathrm{Hol}(V_4)$, such that $\varphi_1(G') = Stab(1_{V_4})$. The extension K/F is Galois with group G'. This is the unique Hopf Galois structure for K/F. We obtain then a unique induced Hopf Galois structure for each G', given by $\varphi : A_4 \hookrightarrow \mathrm{Hol}(V_4 \times C_3)$ such that $\varphi(G') = Stab(\{1_{V_4}\} \times C_3)$. Therefore K/k has four different induced Hopf Galois structures of type $V_4 \times C_3$. We obtain then

$$e(A_4, V_4 \times C_3) \geq 4.$$

2. Groups of order $4p$

p odd prime, G nonabelian group of order $4p$, K/k Galois extension with group G.
G has a unique p-Sylow subgroup H and p 2-Sylow subgroups isomorphic either to C_4 or to $C_2 \times C_2$. Let G' be a 2-Sylow subgroup of G and $F = K^{G'}$.
Since F/k has degree p and G is solvable, F/k is Hopf Galois (Childs 1989). Furthermore, F/k is almost classically Galois and has a unique Hopf Galois structure given by the normal complement H of G' in G.
The number of Hopf Galois structures for Galois extensions with group isomorphic to G' is

	$N_2 \simeq C_4$	$N_2 \simeq C_2 \times C_2$
$G' \simeq C_4$	1	1
$G' \simeq C_2 \times C_2$	3	1

Hence the number of induced Hopf Galois structures of type $H \times N_2$ for K/k is

	Structures $C_4 \times C_p$	Structures $C_2 \times C_2 \times C_p$
2-Sylow subgroup $\simeq C_4$	p	p
2-Sylow subgroup $\simeq C_2 \times C_2$	$3p$	p

These are exactly the numbers of split Hopf Galois structures for K/k of type $C_4 \times C_p$ or $C_2 \times C_2 \times C_p$ (Kohl, 2007).

3. Groups of order pq

G group of order pq, p and q primes, $p > q$, K/k Galois extension with group G.

- If $q \nmid p-1$, pq is a Burnside number and K/k has a unique Hopf Galois structure, the classical Galois one (Byott, 1996).
- If $q \mid p-1$, G is either cyclic or metacyclic $C_p \rtimes C_q$.
  - ▶ If $G \simeq C_{pq}$, there are $2q-1$ different Hopf Galois structures for K/k, the classical one with $N \simeq C_{pq}$ (split) and $2q-2$ structures with $N \simeq C_p \rtimes C_q$ (nonsplit).
  - ▶ If $G \simeq C_p \rtimes C_q$, it has a unique p-Sylow subgroup and p q-Sylow subgroups. Let G' be a q-Sylow subgroup of G and $F = K^{G'}$. Since F/k has prime degree p and G is solvable, F/k is Hopf Galois (Childs, 1989). Furthermore, in this case F/k is almost classically Galois and has a unique Hopf Galois structure. The Galois structure of K/F is also the unique Hopf Galois structure.

    Therefore, for each G', we obtain exactly one induced Hopf Galois structure for K/k and all together we obtain in this way p induced Hopf Galois structures for K/k. This covers all split structures for K/k (Byott, 2004).

In particular, if p is an odd prime and K/k is a dihedral extension of degree $2p$, its Hopf Galois structures are the two given by G and G^{opp} (dihedral type) and the p split structures of type $C_2 \times C_p$ (cyclic type), induced by the structures of K/F and F/k, for $F = K^{G'}$ with G' ranging over the set of complements in G of the cyclic subgroup of order p.