Module theory and Cartan matrices: a result of Schneider and its context

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Abstract

We begin by explaining the background, made up by module theory and a little K-theory. Then we present the Cartan matrix and the Cartan-Brauer triangle in some detail and try to elucidate these concepts by simple examples. Then we state Schneider's result: if R is a complete discrete valution ring which has characteristic 0 and has p in its radical, and H is a finite cocommutative Hopf algebra over R, then the Cartan matrix is nonsingular, and we explain the important consequence in Hopf Galois theory: two projective H-modules are isomorphic as soon as they become isomorphic after base change to the quotient field of R.

0 The rings and modules in play

Let R be a complete discrete valuation ring, rad(R) its radical, $k = R/\mathrm{rad(R)}$ its residue field. We always suppose that R has characteristic 0 and that k has characteristic p. Let K be the field of fractions of R. Then (R, K, k) is a so-called *modular triple*. Let B be an R-algebra, finitely generated and projective as an R-module, and let A be a finite-dimensional k-algebra.

Remark:

1) The objects A and k will often be studied in their own right, but:
2) Whenever B and R are present, it will be understood that $A = k \otimes_R B = B/\mathrm{rad(R)}B$. In this situation, we call B a *lift* of A. The algebras B and A *may* be Hopf algebras over the appropriate rings.

Example: For any finite group D, one can take the data $R = \mathbb{Z}_p$, $k = \mathbb{F}_p$, $B = \mathbb{Z}_p[D]$ and $A = \mathbb{F}_p[D]$.

1 Some module theory

All of the modules that we consider are finitely generated. Let $S \in \{k, A, R, B\}$.

A *projective cover* of a (left) S-module M is a projective (left) S-module P along with a surjective S-module homomorphism $\pi : P \twoheadrightarrow M$ such that $\ker(\pi) \subset \mathrm{rad(P)} = \mathrm{rad(S)}P$. Projective covers exist, and are unique up to (non-unique) isomorphism.

Examples:

1) For any $x \in \mathrm{rad(S)}$, the projection $\pi : S \twoheadrightarrow S/Sx$ is a projective cover.

2) If G is a finite p-group and $S = \mathbb{F}_p[G]$, the augmentation map $\varepsilon : S \twoheadrightarrow \mathbb{F}_p$ is a projective cover.

Now look at the k-algebra A. It is semisimple if and only if all A-modules are projective. In full generality, there are only a finite number of simple A-modules (up to isomorphism of course), say $F_1, \ldots, F_r$, and a finite number of indecomposable projective A-modules, say $U_1, \ldots, U_r$. Note that there are the same number of each:

Proposition 1. *There is a bijection $\{F_i\} \leftrightarrow \{U_i\}$. In one direction, a simple A-module F_i is sent to its projective cover over A. In the other, an indecomposable A-module U_i is sent to $U_i/rad(A)U_i$.*

Moreover, one can say that each U_i occurs as an ideal in A, and as a left A-module, A is the direct sum of indecomposable projectives (possibly with repetitions).

Now enters B (recall that in this situation $A = k \otimes_R B$).

Proposition 2. *Let the modules U_i be defined as above, and let P_i denote the indecomposable projective B-modules. Then there is a bijection $\{P_i\} \leftrightarrow \{U_i\}$. In one direction, an indecomposable B-module P_i is sent to $k \otimes_R P_i$. In the other, an indecomposable A-module U_i is sent to the projective cover of U_i over B.*

The *proof* uses lifting idempotents against a surjective homomorphism with topologically nilpotent kernel.

2 Review of K_0 and G_0

Let $S \in \{k, R, A, B\}$. Recall that

$$K_0(S) = \{\text{projective } S\text{-modules}\}/\text{ short exact sequences},$$

(a short exact sequence $0 \leftarrow P' \to P \to P'' \to 0$ gives the relation $[P] = [P'] + [P'']$), and

$$G_0(S) = \{S\text{-modules of finite length}\}/\text{ short exact sequences}.$$

If S is artinian (for example, if $S = A$) then $G_0(S)$ is a free $\mathbb{Z}$-module on $[F_1], \ldots, [F_r]$ (the classes of the simple A-modules). If $S = A$ (respectively, $S = B$) then $K_0(S)$ is a free $\mathbb{Z}$-module on $[U_1], \ldots, [U_r]$ (respectively, on $[P_1], \ldots, [P_r]$). Therefore we have isomorphisms of abelian groups

$$\begin{array}{ccccc} K_0(B) & \cong & K_0(A) & \cong & G_0(A), \\ [P_i] & \mapsto & [U_i] & \mapsto & [F_i]. \end{array}$$

3 The Cartan matrix

Define $car : K_0(A) \to G_0(A)$ by $[P] \mapsto [P]$ for each A-module P. Note that this is *not* the same map as the one appearing between these two groups at the end of the previous section. More precisely: Let $C = (c_{ij})$ be the representing matrix for car with respect to the $\mathbb{Z}$-bases $\{[U_i]\}$ and $\{[F_j]\}$ of $K_0(A)$ and $G_0(A)$ respectively. Then c_{ij} tells us how often the simple module F_j occurs in a composition series for the indecomposable projective module U_i.

Examples:

1) If A is semisimple then $U_i = F_i$ for all i, so C is the identity matrix.

2) If A is commutative then $A = \bigoplus_{i=1}^{r} A_i$ where each A_i is a local ring with residue field k_i. In this case we have $U_i = 0 \times \cdots \times A_i \times \cdots \times 0$ and $F_i = 0 \times \cdots \times k_i \times \cdots \times 0$, so C is diagonal, with c_{ii} equal to the length of A_i.

3) Let $A = \mathbb{F}_2[S_3]$, where $S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1,\ \tau\sigma = \sigma^2\tau \rangle$. Then

$$A = U_1 \times U_2,$$

where $U_1 = \dfrac{\mathbb{F}_2[\tau]}{(\tau^2 - 1)}$ is an indecomposable, but not simple, A-module, and U_2 is a simple A-module. This decomposition is induced from the decomposition

$$\mathbb{F}_2[\sigma] = \mathbb{F}_2 \oplus \frac{\mathbb{F}_2[\sigma]}{(\sigma^2 + \sigma + 1)}.$$

The simple A-modules are $F_1 = \mathbb{F}_2$ (with trivial action) and $F_2 = U_2$ (appearing in the decomposition above). In this case we find that that

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

4) [Schneider]. Let k have characteristic 2, and let $A = k[x, e]$ with $x^2 = 0$, $e^2 = e$ and $[x, e] = x$. It turns out that $\dim_k(A) = 4$, $\text{rad(A)} = Ax$, and

$$\overline{A} = \frac{A}{\text{rad(A)}} = \frac{k[e]}{(e^2 - e)} = k \times k.$$

There are two simple A-modules (each a copy of k with zero action of x): F_1, which is annihilated by e, and F_2, which is annihilated by $1 - e$. The indecomposable projectives are $U_1 = Ae$ and $U_2 = A(1 - e)$. We have a composition series

$$0 \subset Axe \subset Ae.$$

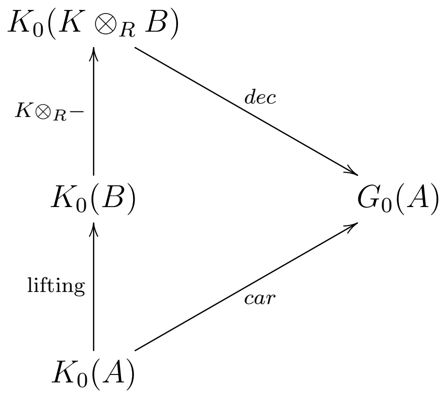
The quotient $\dfrac{Ae}{Axe}$ is annihilated by $1 - e$ and x, so it is isomorphic to F_2. In Axe we have $exe = (xe + x)e = x(e - 1)e = 0$, so Axe is isomorphic to F_1. Continuing in this way, we find that

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

which is singular! (The algebra A in this example is in fact a Hopf algebra, corresponding to the group scheme $\alpha_2 \rtimes \mu_2$.)

4 The Cartan-Brauer Triangle

There is a commutative triangle:



Here the map *dec* is the so-called decomposition homomorphism.

(Added after the conference) The definition of *dec* goes as follows. Given a module M over $K \otimes_R B$, pick a finitely generated R-submodule $\mathcal{L} \subset M$ spanning M over K. (Such submodules $\mathcal{L}$ are called lattices in M.) Define (!) $dec(M) = [k \otimes_R \mathcal{L}]$. The catch is of course that it is not clear whether this is independent of the choice of lattice $\mathcal{L}$. But this can indeed be proved, by a pretty argument which is not too complicated. As soon as one knows that *dec* is well defined, it is easy to prove that it is a homomorphism and that the diagram commutes.

Proposition 3. *If C is nonsingular*[1] *then the following conditional statement is true:*

If P, P' are projective over B and $K \otimes_R P \cong K \otimes_R P'$ as $K \otimes_R B$-modules, then $P \cong P'$.

Proof. We are assuming that *car* is injective; and then the commutative triangle shows that the map $K \otimes_R -$ is also injective. Hence $K \otimes_R P \cong K \otimes_R P'$ implies that the classes of P and P' in $K_0(B)$ are the same. Hence P and P' are stably isomorphic over B. Under our assumptions the Krull dimension of B is 1, so stable isomorphism implies isomorphism. □

The goal of the rest of the talk is now to show that C **is** nonsingular *in the case that B is an R-Hopf algebra.*

5 Schneider's result

Theorem 4. *If B is a cocommutative R-Hopf algebra (finitely generated, projective over R) and $A = k \otimes_R B$, then the Cartan matrix $C = C_A$ is nonsingular.*

[1] that is, if the integer $\det C$ is not zero; we're not saying anything about invertibility

Remark: Schneider also proves that C is symmetric, and that $\det C$ is a power of the characteristic of k if k is a finite field; but we will not deal with these extra statements.

Recall that in example (4) of section 3 we had $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Therefore the corresponding k-algebra A does not come from any Hopf algebra B over R: in Schneider's terminology, A *is not liftable.*

Plan of the Proof:

1) If B is commutative, then we're done. Indeed, since R is complete, B is a finite product of commutative local rings, and then, as said before, C is a nonsingular diagonal matrix.

2) Show the statement for H an order in $K[D]$, the group algebra of a finite group D. This part is modelled on the pre-existing proof in the case that $H = R[D]$. In that case, suppose that $x \in K_0(k[D])$ is in the kernel of car and lies in

$$\bigcup_{\substack{C \leq D \\ C \text{ cyclic}}} \mathrm{ind}_C^D K_0(k[C])$$

(note that each $k[C]$ is commutative). Then one gets that x is zero, using some commutative diagrams and the trivial circumstance that group rings of cyclic groups are commutative (see part 1)). Now use Frobenius functors: these allow us to replace $\bigcup$ with $\sum$. This sum has finite index in $K_0(k[D])$ by Brauer's Induction Theorem. Since $K_0(k[D])$ is $\mathbb{Z}$-torsion free, we have $\ker(car) = 0$.

3) Reduce the general case to the case that $K \otimes_R H$ is a group ring, by a fairly straightforward descent argument.

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