What is a Refined Ramification Break?

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May 23, 2016

Notation for Local Fields

Let K be a local field and let

$$\begin{aligned} v_K &: K \twoheadrightarrow \mathbb{Z} \cup \{\infty\} \\ \mathcal{O}_K &= \{x \in K : v_K(x) \geq 0\} \\ \mathcal{M}_K &= \{x \in K : v_K(x) \geq 1\} \\ \overline{K} &= \mathcal{O}_K/\mathcal{M}_K \text{ perfect, with } \mathrm{char}(\overline{K}) = p \\ |x|_K &= 2^{-v_K(x)}. \end{aligned}$$

Let L/K be a finite totally ramified Galois extension and let $G = \mathrm{Gal}(L/K)$. Then $|G| = [L : K]$ and $\overline{L} \cong \overline{K}$.

A Norm on $K[G]$

Let $\alpha \in K[G]$. Then α is a K-linear operator on L. Therefore it makes sense to define the norm of α:

$$||\alpha||_L = \max\left\{\frac{|\alpha(x)|_L}{|x|_L} : x \in L^\times\right\}$$

Using this norm we define

$$\begin{aligned}\hat{v}_L(\alpha) &= -\log_2 ||\alpha||_L \\ &= \min\{v_L(\alpha(x)) - v_L(x) : x \in L^\times\} \\ &\in \mathbb{Z} \cup \{\infty\}.\end{aligned}$$

Let I be an ideal in $K[G]$. One can define a norm on $K[G]/I$ by setting

$$||\alpha + I||_L = \min\{||\alpha'||_L : \alpha' \in \alpha + I\}.$$

Pseudo-valuations

For $\alpha, \beta \in K[G]$ we have

$$\hat{v}_L(\alpha) = \infty \Leftrightarrow \alpha = 0$$
$$\hat{v}_L(\alpha\beta) \geq \hat{v}_L(\alpha) + \hat{v}_L(\beta)$$
$$\hat{v}_L(\alpha + \beta) \geq \min\{\hat{v}_L(\alpha), \hat{v}_L(\beta)\}.$$

We say that $\hat{v}_L$ is a *pseudo-valuation* on $K[G]$.

If I is an ideal in $K[G]$ we get a pseudo-valuation on $K[G]/I$ by setting

$$\begin{aligned}\hat{v}_L(\alpha + I) &= -\log_2 ||\alpha + I||_L \\ &= \max\{\hat{v}_L(\alpha') : \alpha' \in \alpha + I\}\end{aligned}$$

for $\alpha + I \in K[G]/I$.

Ramification Breaks

For $a \in \mathbb{N} \cup \{0\}$, we define the ath lower ramification subgroup of $G = \mathrm{Gal}(L/K)$ to be

$$G_a = \left\{ \sigma \in G : v_L \left(\frac{\sigma(\pi_L) - \pi_L}{\pi_L} \right) \geq a \right\}.$$

Say $b \in \mathbb{N} \cup \{0\}$ is a (lower) ramification break of L/K if $G_b \neq G_{b+1}$.

Alternatively, we have

$$\begin{aligned} G_a &= \{\sigma \in G : \hat{v}_L(\sigma - 1) \geq a\} \\ &= \{\sigma \in G : ||\sigma - 1||_L \leq 2^{-a}\}. \end{aligned}$$

Thus b is a ramification break of L/K if and only if $\hat{v}_L(\sigma - 1) = b$ for some $\sigma \in G$.

Extending Ramification Data

Write $[L : K] = n = cp^r$, with $p \nmid c$. Then the number of positive ramification breaks of L/K is at most r.

If L/K has fewer than r positive ramification breaks then L/K is (in some sense) degenerate.

This occurs if and only if there is $b \geq 1$ such that G_b/G_{b+1} is an elementary abelian p-group of rank > 1.

Attempts have been made to supply the "missing" ramification data:

- Indices of inseparability (Fried, Heiermann)
- Refined ramification breaks (Byott-Elder)

Defining New Ramification Breaks

Following Byott-Elder, we will attempt to define new ramification breaks by adding $\mathcal{O}_K$-coefficients to G.

Can we define the missing breaks as values of $v_L\left(\frac{\alpha(x)}{x}\right)$ for some $\alpha \in \mathcal{O}_K[G]$ and $x \in L^\times$?

Alternatively, can we define refined breaks to be values of $\hat{v}_L(\alpha)$ for $\alpha \in \mathcal{O}_K[G]$?

In either case we recover the ordinary ramification breaks by letting $\alpha = \sigma - 1$ with $\sigma \in G$.

Unfortunately, these definitions give us infinitely many breaks.

Getting r Breaks

To get exactly r breaks, do one of the following:

1. Restrict the allowable choices of α (and x).
2. Define breaks to be values of $\hat{v}_L(\alpha + I)$ for some ideal $I \subset K[G]$.
3. Combine 1 and 2 somehow.

Byott-Elder use method 3: Breaks are values of
$v_L\left(\frac{(\alpha - 1)x}{x}\right)$ for certain $x \in L$ and $\alpha \in \mathcal{O}_K[G]/I$.

They focus on the case where G has a single break $b \geq 1$.

Thus G is an elementary abelian p-group of rank r.
We assume $r \geq 2$.

They need to raise elements of G to $\overline{K}$ powers?!

Truncated Powers

Suppose $\mathrm{char}(K) = 0$. Then for $\psi(X) \in XK[[X]]$ and $c \in K$ we can define

$$(1+\psi(X))^c = \sum_{n=0}^{\infty} \binom{c}{n} \psi(X)^n, \text{ where}$$
$$\binom{c}{n} = \frac{c(c-1)(c-2)\dots(c-(n-1))}{n!}.$$

For an arbitrary local field K, Byott and Elder defined the "truncated cth power" of $1+\psi(X)$ to be

$$(1+\psi(X))^{[c]} = \sum_{n=0}^{p-1} \binom{c}{n} \psi(X)^n.$$

Multiplicative $\mathcal{O}_K$-Module Structures

Suppose $c \in \mathcal{O}_K$. Then $e_c(X) = (1+X)^{[c]}$ lies in $\mathcal{O}_K[X]$.

Let $J_{\mathcal{O}_K} = (\sigma - 1 : \sigma \in G)$ be the augmentation ideal of $\mathcal{O}_K[G]$.

For $\alpha \in 1 + J_{\mathcal{O}_K}$ define $\alpha^{[c]} = e_c(\alpha - 1)$. Then $\alpha^{[c]} \in \mathcal{O}_K[G]$.

The scalar multiplication $c \cdot \alpha = \alpha^{[c]}$ does not make the multiplicative group $(1 + J_{\mathcal{O}_K})^\times$ an $\mathcal{O}_K$-module.

But it does make the quotient $(1 + J_{\mathcal{O}_K})^\times / (1 + J_{\mathcal{O}_K}^p)^\times$ an $\mathcal{O}_K$-module.

Since $(1 + J_{\mathcal{O}_K})^\times / (1 + J_{\mathcal{O}_K}^p)^\times$ is killed by p, it is a module over $\mathcal{O}_K / p\mathcal{O}_K$.

Since $\overline{K}$ can be embedded into $\mathcal{O}_K / p\mathcal{O}_K$, we see that $(1 + J_{\mathcal{O}_K})^\times / (1 + J_{\mathcal{O}_K}^p)^\times$ is a vector space over $\overline{K}$.

Refined Ramification Breaks (Byott-Elder)

Suppose $G = \mathrm{Gal}(L/K)$ has a single ramification break $b \geq 1$. Then G is an elementary abelian p-group of rank r for some $r \geq 1$.

Let $G^{[\overline{K}]}$ denote the $\overline{K}$-span of the image of G in $(1 + J_{\mathcal{O}_K})/(1 + J_{\mathcal{O}_K}^p)$. Then $\dim_{\overline{K}}(G^{[\overline{K}]}) = r$.

For $\alpha + J_{\mathcal{O}_K}^p \in G^{[\overline{K}]}$ and $x \in L^\times$ define

$$i_x(\alpha + J_{\mathcal{O}_K}^p) = \max\{v_L(\alpha'(x) - x) : \alpha' \in \alpha + J_{\mathcal{O}_K}^p\}.$$

Suppose $v_L(x) = b$. We say that a is a *refined ramification break* of L/K (with respect to x) if $a = i_x(\alpha + J_{\mathcal{O}_K}^p) - v_L(x)$ for some $\alpha + J_{\mathcal{O}_K}^p \in G^{[\overline{K}]}$.

What is Known about Refined Breaks

Assume that L/K has a single ordinary ramification break b and $G \cong C_p^r$. Then

- b is a refined break of L/K.
- Every refined break a of L/K satisfies $a \geq b$.
- The number of refined breaks of L/K is r.
- If $\mathrm{char}(K) = 0$, $r = 2$, and K contains a primitive pth root of unity, the refined breaks can be computed in terms of Kummer theory (Byott-Elder).
- If $\mathrm{char}(K) = p$ and $r = 2$, the refined breaks can be computed in terms of Artin-Schreier theory (Elder-Keating).
- In both rank-2 settings the values of the refined breaks do not depend on the choice of x, as long as $v_L(x) = b$.

Extended Ramification Breaks

Assume L/K has a single ramification break b and $G \cong C_p^r$.

Define the "extended ramification breaks" of L/K to be the positive integers of the form $e = \hat{v}_L(\alpha - 1 + J_{\mathcal{O}_K}^p)$ with $\alpha + J_{\mathcal{O}_K}^p \in G^{[\overline{K}]}$.

This avoids the choice of a special $x \in L$, so the extended ramification breaks of L/K are well-defined.

If $r = 2$ the extended breaks of L/K are the same as the refined breaks of L/K.

It's easy to see that L/K has at most r distinct extended breaks. It's not known whether there must be exactly r distinct extended breaks.

Delicate Ramification Breaks

The map $\sigma \mapsto \sigma - 1$ induces an isomorphism from G/G' to $J_{\mathbb{Z}}/J_{\mathbb{Z}}^2$.

Hence if G is abelian then $G \cong G/G' \cong J_{\mathbb{Z}}/J_{\mathbb{Z}}^2$.

Suppose $\mathrm{char}(K) = 0$, and let K_0 be the subfield of K such that K/K_0 is a totally ramified extension of degree $v_K(p)$. Then $v_{K_0}(p) = 1$ and $\overline{K}_0 \cong \overline{K}$.

We get $G \otimes_{\mathbb{Z}} \mathcal{O}_{K_0} \cong J_{\mathcal{O}_{K_0}}/J_{\mathcal{O}_{K_0}}^2$.

Say d is a *delicate ramification break* of L/K if $d = \hat{v}_L(\alpha + J_{\mathcal{O}_{K_0}}^2)$ for some $\alpha \in J_{\mathcal{O}_{K_0}}$.

If $G \cong C_p^2$ and L/K has a single (ordinary) ramification break then the delicate breaks of L/K are the same as the refined breaks of L/K.

Pros and Cons: Refined Breaks

$a = \max\{v_L(\alpha'(x) - x) : \alpha' \in \alpha + J^p_{\mathcal{O}_K}\}$ with

$\alpha + J^p_{\mathcal{O}_K} \in G^{[\overline{K}]}$ and $v_L(x) = b$

Good:

- If L/K has a single break b and $G \cong C_p^r$ then there are r refined breaks, including b.
- When $r = 2$ it gives information about $\mathcal{O}_L$ as an $\mathcal{O}_{K_0}[G]$-module.

Bad:

- Are these breaks well-defined invariants of L/K, or do they depend on the choice of x?
- Uses truncated powers, which seems somewhat arbitrary.
- Refined breaks are only defined for elementary abelian extensions with a single ordinary break.

Pros and Cons: Extended Breaks

$e = \hat{v}_L(\alpha - 1 + J^p_{\mathcal{O}_K})$ with $\alpha + J^p_{\mathcal{O}_K} \in G^{[\overline{K}]}$

Good:

- Avoids arbitrary choice of x.

Bad:

- Uses truncated powers.
- Only defined for elementary abelian extensions with a single ordinary break.

Pros and Cons: Delicate Breaks

$d = \hat{v}_L(\alpha + J^2_{\mathcal{O}_{K_0}})$ with $\alpha \in J_{\mathcal{O}_{K_0}}$

Good:

- Melds G with $\overline{K}$ nicely.
- Applies to some extensions which are not elementary abelian.
- Avoids truncated powers and choice of x.

Bad:

- Is every ordinary ramification break of L/K a delicate break?
- This method does not apply to fields of characteristic p, or to nonabelian extensions.