Dieudonné module theory, part III: applications

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Outline

Recall:

Let k be perfect, char $k = p > 2$.

Let H be a (finite, abelian) local-local k-Hopf algebra.

Then $M := D_*(H)$ is a finite length $E := W[F, V]$-module killed by a power of F and V.

If $m \in M$ is killed by F^r then the corresponding element $t \in H$ satisfies $t^{p^r} = 0$.

The action of V on M gives the comultiplication on H.

Today, we will see how Dieudonné modules provide us (at least me) with a deeper understanding of Hopf algebras.

Outline

Recall that a k-Hopf algebra is monogenic if it generated as a k-algebra by a single element.

In our local-local case, $H = k[t]/(t^{p^n})$ for some n.

Thus, the algebra structure is known. What about the coalgebra structure?

$H = k[t]/(t^{p^n})$

Let $M = D_*(H)$, and pick $m \in M$.

Then

$$0 = T_0 = (T_m)^{p^n} = T_{F^n m},$$

and since $t^{p^{n-1}} \neq 0$ we see that there exists an $x \in M$ such that

$$F^{n-1}x \neq 0,\ F^n x = 0.$$

$H = k[t]/(t^{p^n}),\ F^n x = 0,\ x \in M$

We will show that M is generated by x, so M is a *cyclic* Dieudonné module.

Let $M' = Ex \subseteq M$.

We have a chain of E-modules (also W-modules)

$$0 = F^n M' \subset F^{n-1} M' \subset \cdots \subset FM' \subset M' \subseteq M.$$

Claim $F^i M' \neq F^{i-1} M'$ for all $1 \leq i \leq n$.

$0 = F^n M' \subset F^{n-1} M' \subset \cdots \subset FM' \subset M' \subseteq M$

Claim $F^i M' \neq F^{i-1} M'$ for all $1 \leq i \leq n$.

Proof: If $F^{i-1} M' = F^i M'$ then $F^{i-1} x = F^i ex$ for some $e \in E$.

But $F^i ex$ is killed by F^{n-i},so

$$F^{n-i} F^{i-1} x = F^{n-i} F^i ex = 0 \Rightarrow F^{n-1} x = 0,$$

which is a contradiction.

$H = k[t]/(t^{p^n}),\ F^n x = 0,\ x \in M$

$$0 = F^n M' \subset F^{n-1} M' \subset \cdots \subset FM' \subset M' \subseteq M.$$

Since $\dim_k H = p^n$, M has length n over W.

Thus, any chain of W-submodules of M can have length at most n.

Since all the $F^i M'$ are distinct, it follows that $M' = M$ and M is cyclic.

$$0 = F^n M \subset F^{n-1} M \subset \cdots \subset FM \subset M.$$

What about Vx?

Consider the chain

$$0 = F^n M \subset F^{n-1} M \subset \cdots \subset FM \subseteq FM + VM \subseteq M.$$

Again, by length considerations it follows that either

1. $FM + VM = FM$, in which case $VM \subseteq FM$. In particular, $Vx \in FM$.
2. $FM + VM = M$, in which case $x = Fe_1 x + Ve_2 x,\ e_1, e_2 \in E$.

We claim this second case cannot occur.

The case $FM + VM = M$, $x = Fe_1x + Ve_2x$

Let N be the smallest positive integer such that $V^{N+1}x = 0$. Then

$$V^N x = V^N Fe_1 x + V^{N+1} e_2 x = V^N Fe_1 x,$$

so $V^N(1 - e_1^\sigma F)x = 0$ where e_1^σ is e_1 with σ applied to all the W-coefficients.
Thus, $(1 - eF)V^N x = 0$ where $e = e_1^{\sigma^{N-1}}$.
Applying $(1 + eF + (ee^\sigma)F^2 + \cdots + (ee^\sigma \cdots e^{\sigma^{n-1}})F^{n-1})$ to both sides:

$$\begin{aligned}
(1 + eF + (ee^\sigma)F^2 + \cdots + (ee^\sigma \cdots e^{\sigma^{n-1}})F^{n-1}))(1 - eF)V^N x &= 0 \\
(1 - (ee^\sigma \cdots e^{\sigma^n})F^n)V^N x &= 0 \\
V^N x - (ee^\sigma \cdots e^{\sigma^n})V^N F^n x &= 0 \\
V^N x &= 0
\end{aligned}$$

(recall $F^n x = 0$ and $FV = VF$), which is a contradiction.

$H = k[t]/(t^{p^n}),\ F^n x = 0,\ Vx \in FM$

We can write $Vx = f(F)F^r x$ for some $r \le n$ and $f(F) \in W[F] \subset E$ with nonzero constant term.

For brevity, write f for $f[F]$.

However,

$$px = FVx = fF^{r+1}x,$$

and since any $w = (w_0, w_1, w_2, \dots) \in W$ decomposes as

$$(w_0, w_1, w_2, \dots) = \sum_{i=0}^{\infty} p^i (w_i^{\sigma^{-i}}, 0, 0, \dots)$$

we may assume $f \in k[F]$.

Furthermore, we may abuse notation and assume $f \in \left(k[F]/(F^{n-r})\right)^\times$.

Exercise 1. Show that $r > 0$. (bf Hint. It suffices to show that M/FM is not a Dieudonné module if $r = 0$.)

Thus,

$$M = E/E(F^n, fF^r - V),\ 1 \le r \le n,\ f \in \left(k[F]/(F^{n-r})\right)^\times.$$

This classification breaks up naturally into two classes:

1. $r = n$, giving $M = E/E(F^n, V)$.
2. $1 \le r < n$.

Exercise 2. As explicitly as possible (which may not be very explicit), write out $\Delta(t)$ in both classes.

Exercise 3. Show that if k contains $\mathbb{F}_{p^{p^2}}$ then

$$E/E(F^3, F^2 - F - V) = E/E(F^3, F - V).$$

Exercise 4. Show that if $k = \mathbb{F}_{p^2}$ then

$$E/E(F^3, F^2 - F - V) \neq E/E(F^3, F - V).$$

Question. When is

$$M := E/E(F^n, fF^r - V) \cong E/E(F^{n'}, f'F^{r'} - V) =: M'?$$

Exercise 5. Suppose $r = n$. Show that $M \cong M'$ if and only if $r' = n' = n$.

Thus, $E/E(F^n, V)$ is in “a class by itself”.

Exercise 6. Suppose $r < n$. Show that if $M \cong M'$ then $n = n'$ and $r = r'$

In general, the converse to the above exercise is not true, as we have already seen.

$M := E/E(F^n, fF^r - V) \cong E/E(F^n, f'F^r - V) =: M'$?

Let $x \in M, y \in M'$ be the image of 1 under the canonical maps $E \to M,\ E \to M'$ respectively.
Suppose $\phi : M \to M'$ is an E-module homomorphism.
Then $\phi(x) = gy$ for some $g \in k[F]/(F^n)$ and

$$\phi(fF^r x) = fF^r\phi(x) = fF^r gy = fg^{(p^r)}F^r y$$

$$\phi(Vx) = V\phi(x) = Vgy = g^{(p^{-1})}Vy = g^{(p^{-1})}f'F^r y,$$

where $g^{(\ell)}$ raises each coefficient to the ℓ^{th} power.
Thus:

Proposition

$E/E(F^n, fF^r - V) \cong E/E(F^n, f''F^r - V)$ if and only if there exists a $g \in \left(k[F]/(F^n)\right)^\times$ such that

$$fg^{(p^r)}F^r y = g^{(p^{-1})}f'F^r y$$

Exercise 7. Pick $r > 0$, and suppose $k \subseteq \mathbb{F}_{p^{r+1}}$. For $n > r$ partition the set $\{E/E(F^n, fF^r - V) : f \in \left(k[F]/(F^n)\right)^\times\}$ into isomorphism classes.

Exercise 8. Count the number of monogenic local-local $\mathbb{F}_p$-Hopf algebras of rank p^n.

Exercise 9. Suppose k is algebraically closed. For fixed n, r partition the set $\{E/E(F^n, fF^r - V) : f \in \left(k[F]/(F^n)\right)^\times\}$ into isomorphism classes.

Exercise 10. Continuing with k algebraically closed, count the number of monogenic local-local k-Hopf algebras of rank p^n.

Outline

A local-local k-Hopf algebra is said to be *bigenic* if it can be generated as a k-algebra by two elements.

Note. This definition includes monogenic Hopf algebras.

For simplicity, suppose k is algebraically closed.

Then (spoiler alert) $M = E/E(F^n, F^r - V)$ for some $1 \leq r \leq n$.

Questions.

1. Can we find all bigenic Hopf algebras?
2. Can we easily determine which bigenic Hopf algebras are also monogenic?

Fundamental observation 1. Bigenic Hopf algebras can be realized as extensions of monogenic Hopf algebras.

Fundamental observation 2. The category of E-modules allows for extensions.

Thus, we can find bigenic Hopf algebras if we can compute $\mathrm{Ext}^1(M, M')$, where

$$M = E/E(F^n, F^r - V), \; M' = E/E(F^{n'}, F^{r'} - V).$$

$M = E/E(F^n, F^r - V),\ M' = E/E(F^{n'}, F^{r'} - V)$

A projective presentation for M is

$$0 \to (EF^n + E(F^r - V)) \to E \to M \to 0,$$

giving rise to

$$\mathrm{Hom}_E(E, M') \to \mathrm{Hom}_E(EF^n + E(F^r - V), M') \to \mathrm{Ext}^1(M, M') \to 0.$$

In a manner similar to Monday's talk, $\mathrm{Hom}_E(E, M') \cong M'$.

Thus, the trickiest part is the computation of

$$\mathrm{Hom}_E(EF^n + E(F^r - V), M').$$

$\mathrm{Hom}_E(EF^n + E(F^r - V), M')$ (Sketch)

Let $z \in M'$ be the image of $1 \in E$ under the canonical map $E \to M'$.

Let $\phi : EF^n + E(F^r - V) \to M'$ be an E-module map.

Then $\phi \mapsto (\phi(F^n), \phi(F^r - V))$ defines a map

$$\mathrm{Hom}_E(EF^n + E(F^r - V), M') \to M' \times M',$$

and say the image of ϕ is (gz, hz) for $g, h \in k[F]$.

Since ϕ is well-defined on $F^n(E^r - V)$, we need

$$h^{(p^n)} F^n z = g^{(p^r)} F^r z - g^{(p^{-1})} F^{r'} z,$$

conversely any choice of g, h as above corresponds to some ϕ.

Also, the set S of all pairs $(gz, hz) \in M'$ satisfying the above conditions is a subgroup of $M' \times M'$.

$\mathrm{Hom}_E(E, M') \to \mathrm{Hom}_E(EF^n + E(F^r - V), M')$

The map $M' \to S$ corresponding to
$\mathrm{Hom}_E(E, M') \to \mathrm{Hom}_E(EF^n + E(F^r - V), M')$ is

$$fz \mapsto (F^n fx, (F^r - V)fz) = (f^{(p^n)}F^n z, (f^{(p^r)}F^r - f^{(p^{-1})}F^{r'})z).$$

Let S_0 be the image of this map.

Then $\mathrm{Ext}^1(M, M') \cong S/S_0$.

From this we can get

Theorem

Every extension of M by M' is of the form $M_{g,h}$, where $M_{g,h}$ is generated by two elements x, y such that

$$F^n x = gy, \ (F^r - V)x = hy, \ F^{n'}y = 0, \ (F^{r'} - V)y = 0.$$

A k-basis for $M_{g,h}$ is $\{x, Fx, \ldots, F^{n-1}x, y, Fy, \ldots, F^{n'-1}y\}$.

A technical note

Theorem

Every extension of M by M' is of the form $M_{g,h}$, where $M_{g,h}$ is generated by two elements x, y such that

$$F^n x = gy,\ (F^r - V)x = hy,\ F^{n'} y = 0,\ (F^{r'} - V)y = 0.$$

A k-basis for $M_{g,h}$ is $\{x, Fx, \ldots, F^{n-1}x, y, Fy, \ldots, F^{n'-1}y\}$.

$M_{g,h}$ is an E-module.

Is it necessarily a Dieudonné module?

Yes.

Exercise 11. Use the relations above to show that $M_{g,h}$ is killed by a power of F and V.

Theorem

Every extension of M by M' is of the form $M_{g,h}$, where $M_{g,h}$ is generated by two elements x, y such that

$$F^n x = gy,\ (F^r - V)x = hy,\ F^{n'} y = 0,\ (F^{r'} - V)y = 0.$$

A k-basis for $M_{g,h}$ is $\{x, Fx, \ldots, F^{n-1}x, y, Fy, \ldots, F^{n'-1}y\}$.

Exercise 12. Show that H is monogenic if and only if $g \in k[F]$ has nonzero constant term.

Exercise 13. More generally, suppose $g \in F^v f[F] \setminus F^{v+1}k[F]$. Show that $H \cong k[t_1, t_2]/(t_1^{p^{n+n'-v}}, t_2^{p^v})$.

Exercise 14. Suppose $n = r,\ n' = r',\ g = F^i,\ h = F^j,\ n' \le i < n,\ j < n'$. Give both the algebra and the coalgebra structure on the bigenic Hopf algebra.

Outline

Return to k any perfect field of characteristic $p > 2$.

A Hopf algebra H is said to be of *height one* if $x^p = 0$ for all $x \in H^+ := \ker \varepsilon$.

Obvious example: $H = k[t]/(t^p),\ \Delta(t) = t \otimes 1 + 1 \otimes t$.

Objective. Use Dieudonné modules to classify all height one (local-local) k-Hopf algebras.

Let H be height one, so $H = k[t_1, \ldots, t_n]/(t_1^p, \ldots, t_n^p)$, and let $M = D_*(H)$.

Pick $m \in M$. Then $T_m \in H = k \oplus H^+$, so $T_m = a + h;\ a \in k, h \in H^+$.

Then

$$T_{Fm} = (T_m)^p = (a+h)^p = a^p + h^p = a^p.$$

Pick n such that $F^n M = 0$. Then

$$a^{p^n} = (T_m)^{p^n} = T_{F^n m} = T_0 = 0,$$

so $a = 0$, which implies $T_{Fm} = 0$, so $Fm = 0$.

Thus, $FM = 0$.

H Height 1 $\Rightarrow FM = 0$

Now suppose $FM = 0$.

Pick $t \in H^+$. Then

$$t = f(T_{m_1}, T_{m_2}, \ldots, T_{m_s})$$

for some $m_i \in M$ and polynomial f with no constant term.

Then

$$t^p = \left(f(T_{m_1}, T_{m_2}, \ldots, T_{m_s})\right)^p = 0,$$

so $t^p = 0$.

Thus H is height one if and only if $F\left(D_*(H)\right) = 0$.

If $FM = 0$ then M can be viewed as a $k[V]$-module.

$k[V]$ is almost a PID (where "almost" = "is" when $k = \mathbb{F}_p$), and has a module classification similar to that of a PID.

It turns out that M has a unique decomposition

$$M \cong E/E(F, V^{n_1}) \oplus E/E(F, V^{n_2}) \oplus \cdots \oplus E/E(F, V^{n_j})$$

where $n_1 \geq n_2 \geq \cdots \geq n_j$.

Exercise 15. Write out all Hopf algebras of height one and rank p^4. Be an explicit as you can, including both the algebra and coalgebra structure.

Exercise 16. Determine the number of height one Hopf algebras of rank p^6.

Outline

Question. Can Dieudonné module theory, or related theories, be used to classify other Hopf algebras?

Yes.

- A Dieudonné module theory exists which classifies *all* p-power rank cocommutative k-Hopf algebras.

  It requires replacing Witt vectors with something called Witt covectors, and the nature of the correspondence is a bit different.

  See J.-M. Fontaine, Groups finis commutatifs sur les vecteurs de Witt, C.R. Acad. Sci. Paris 280 (1975), 1423–1425 (no proofs given).

Question. Can Dieudonné module theory, or related theories, be used to classify other Hopf algebras?

Yes.

- A Dieudonné module theory exists which classifies p-divisible groups or formal groups.

  In the local-local case, this theory is compatible with ours, allowing for smooth resolutions.

  See:

  - A. Grothendieck, Groupes de Barsotti-Tate et Cristaux de Dieudonné, Les Presses de L'Uinversité de Montreal, 1974.
  - J.-M. Fontaine, Sur la construction du module de Dieudonné d'un groups formel, C.R. Acad. Sci. Paris 280 (1975), 1273–1276 (no proofs given). This is the full, Witt covector case.

Question. Can Dieudonné module theory, or related theory, be used to classify Hopf algebras over different base rings?

Yes.for example:

- Over $R = W(k)$ an unramified extension of $\mathbb{Z}_p$: use "Finite Honda Systems", pairs (M, L) where M is the Dieudonné module for the Hopf algebra over k and L encodes data to lift from k to $W(k)$. This correspondence can be made explicit.
  J. M. Fontaine, Groupes finis commutatifs sur les vecteurs de Witt, C. R. Acad. Sci. Paris 280 (1975), 1423-1425.

- Over R a totally ramified extension of $W(k)$, $e(R/W(k)) \leq p - 1$: use Conrad's systems. The correspondence is a bit less evident.
  B. Conrad, Finite group schemes with bases over low ramification, Compositio Mathematica 119 (1999) 239-320.

- $R/W(k)$ totally ramified: use one of the following.
  - Breuil modules. Very nice for group schemes killed by p (so Hopf algebras killed by $[p]$), although the exact correspondence is usually a mystery to me.
    C. Breuil, Groupes p-divisibles, groupes finis et modules filtrés, Ann. Math 152 (2000) 489-549
  - Breuil-Kisin modules. Better for Hopf algebras not killed by $[p]$. Although the key to understanding the correspondence is by translating from Breuil-Kisin modules to Breuil modules.
    M. Kisin, Moduli of finite flat group schemes, and modularity. Ann. Math. (2) 170(3) (2009):1085–1180.
    Note. Suitably defined, Breuil-Kisin modules work over complete regular local rings R (including the case char $R = p$:
    E. Lau, Frames and finite groups schemes over complete regular local rings, Documenta Math. 15 (2010), 545–569).
  - Dieudonné windows, displays, frames, etc. I don't know much about these. To get you started:
    https://www.math.uni-bielefeld.de/ zink/CFTpaper.pdf.
  - Crystalline Dieudonné module theory. Probably not worth the effort.

Thank you.