The Exercises From Day 2

Tuesday, May 24, 2016

1. Prove:

$$S_1((X_0, X_1); (Y_0, Y_1)) = X_1 + Y_1 - \frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i} X_0^i Y_0^{p-i}$$
$$P_1((X_0, X_1); (Y_0, Y_1)) = X_0^p Y_1 + X_1^p Y_0 + pX_1 Y_1.$$

2. Show that W is a ring of characteristic zero.
3. Show that W is an integral domain.
4. Show that $(1, 0, 0, \dots)$ is the multiplicative identity of W.
5. Show that $W(\mathbb{F}_p) \cong \mathbb{Z}_p$.
6. Show that $W(\mathbb{F}_{p^n})$ is the unramified extension of $\mathbb{Z}_p$ of degree n.
7. Show that the element $(0, 1, 0, 0 \dots) \in W$ acts as mult. by p.
8. Show that $p^n W$ is an ideal of W.
9. Let $W_n = (w_0, w_1, \dots, w_{n-1})$. Show that W_n is a ring with operations induced from W.
10. Show that $W/p^n W \cong W_n$. In particular, $W_0 \cong k$.
11. Show that $FV = VF = p$ (multiplication by p).
12. Show that F, V both act freely on W, only F acts transitively.
13. Show that if $w \in p^n W$ then $Fw, Vw \in p^n W$. Thus, F and V make sense on W_n as well.
14. Show that W_n is annihilated by V^{n+1}.
15. Show that any $w = (w_0, w_1, w_2, \dots) \in W$ decomposes as

$$(w_0, w_1, w_2, \dots) = \sum_{i=0}^{\infty} p^i (w_i^{p^{-i}}, 0, 0, \dots).$$

16. Show that the N picked such that $V^{N+1}M = 0$ in the Dieudonné module need not be minimal.

17. Let M be a Dieudonné module. Show that $T_0 = 0$.
18. Let $M = D_*(H) = kx$. Prove H is generated as a k-algebra by t, where $t = T_x$.

For the next five problems, let $M = E/(F^m, V^n) = D_*(H)$.

19. What is pM?
20. Exhibit a k-basis for M.
21. Prove that
$$H = k[t_1, \ldots, t_n]/(t_1^{p^m}, \ldots, t_n^{p^m})$$
22. Write out the comultiplication (in terms of Witt vector addition).
23. Show that $\mathrm{Spec}(H) = W_n^m$.

For the next four problems, write out the Hopf algebra for each of the following. Be as explicit as you can.

24. $M = E/(F^2, F - V)$
25. $M = E/E(F^2, V^2)$
26. $M = E/E(F^2, p, V^2)$
27. $M = Ex + Ey,\ F^4x = 0, F^3y = 0, Vx = F^2y, V^2x = 0, Vy = 0.$

For the next three problems, find the Dieudonné module for each of the following.

28. $H = k[t]/(t^{p^4}),\ t$ primitive.
29. $H = k[t_1, t_2]/(t_1^{p^3}, t_2^{p^2}),\ t_1$ primitive and
$$\Delta(t_2) = t_2 \otimes 1 + 1 \otimes t_2 + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t_1^{pi} \otimes t_1^{p(p-i)}.$$
30. $H = k[t_1, t_2]/(t_1^{p^3}, t_2^{p^2}),\ t_1$ primitive and
$$\Delta(t_2) = t_2 \otimes 1 + 1 \otimes t_2 + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t_1^{p^2 i} \otimes t_1^{p^2(p-i)}.$$

For the final three problems, let $H = \mathbb{F}_p[t]/(t^{p^3}),\ M^* = E/E(V^3, V^2 - F)$

31. Using M^*, give the algebra structure for H^*. (**Hint.** It requires two generators.)
32. Using M^*, give the coalgebra structure for H^*.
33. What changes if $k \neq \mathbb{F}_p$?