Extensions of classical Hopf-Galois structures

Paul Truman

Keele University, UK

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Hopf-Galois structures and fixed fields

- Let L/K be a Galois extension of fields with Galois group G.
- Suppose that $L[N]^G$ gives a Hopf-Galois structure on L/K, and that N has a G-stable subgroup P.
- Then $L[P]^G$ is a sub-Hopf algebra of $L[N]^G$. Its fixed field is

$$L^P = \{x \in L \mid z \cdot x = \varepsilon(z)x \text{ for all } z \in L[P]^G\}.$$

- $L^P \otimes_K L[P]^G$ gives a Hopf-Galois structure on L/L^P.

Theorem (Koch, Kohl, T, Underwood)

In the situation described above:

- $J = \mathrm{Gal}(L/L^P) = P \cdot 1_G$;
- If $P \triangleleft N$ and $J \triangleleft G$ then N/P acts regularly on G/J, and $L^P[N/P]^{G/J}$ gives a Hopf-Galois structure on L^P/K.

Turning the situation around

Question

If we are given:

- a normal subgroup J of G;
- Hopf-Galois structures on L/L^J and L^J/K,

can these Hopf-Galois structures be combined to give a Hopf-Galois structure on L/K?

- The Hopf-Galois structure on L/K should have the correct substructure and quotient.

Equivalently...

- Thanks to Greither and Pareigis, the question on the previous slide is equivalent to:

Question

If we are given:

- a normal subgroup J of G;
- a regular subgroup of $\mathrm{Perm}(J)$ normalized by $\lambda(J)$ and a regular subgroup of $\mathrm{Perm}(G/J)$ normalized by $\lambda(G/J)$,

can these be combined to give a regular subgroup of $\mathrm{Perm}(G)$ normalized by $\lambda(G)$?

- The regular subgroup of $\mathrm{Perm}(G)$ ought to have the correct normal subgroup and quotient.

The work of Crespo et al.

Theorem (Crespo, Rio, Vela, 2016)

Let F/K be a subextension of L/K, and suppose that

- $\mathrm{Gal}(L/F)$ has a normal complement in G;
- L/F and F/K admit Hopf-Galois structures with underlying groups N, N' respectively.

Then L/K admits a Hopf-Galois structure with underlying group $N \times N'$.

- We are interested in the case $\mathrm{Gal}(L/F) \triangleleft G$. If in addition $\mathrm{Gal}(L/F)$ has a normal complement C in G then $G = \mathrm{Gal}(L/F) \times C$.

Group Extensions

- Let J, Q be abelian groups.
- A group G is called an extension of J by Q if we have a short exact sequence:

$$1 \longrightarrow J \longrightarrow G \underset{\pi}{\overset{\ell}{\leftrightarrows}} Q \longrightarrow 1$$

- Fix a function $\ell : Q \to G$ such that $\pi(\ell(x)) = x$ for all $x \in Q$.
- There is a homomorphism $\theta : Q \to \mathrm{Aut}(J)$. Write $x * j = \theta(x)[j]$.
- There is a function $g : Q \times Q \to J$ such that
  - $g(x, 1) = g(1, x) = 1$ for all $x \in Q$;
  - $g(x, y)g(xy, z) = x * g(y, z)g(x, yz)$ for all $x, y, z \in Q$.

  (That is, a 2-cocycle for $*$)

  Changing ℓ changes this by a 2-coboundary.

Conversely . . .

- Suppose we are given
  - a homomorphism $\theta : Q \to \mathrm{Aut}(J)$,
  - a 2-cocycle g for the corresponding action $*$ of Q on J.
- Let $G = G(J, Q, \theta, g)$ be the set $\{(j, q) \mid j \in J, q \in Q\}$ with the multiplication

$$(j, q)(j', q') = (j(q * j')g(q, q'), qq').$$

- Then G is an extension of J by Q.

Theorem (Schreier, 1926)

There is a bijection between equivalence classes of extensions based on J, Q, θ and the group $H^2(Q, J, \theta)$.

Restrictions and projections of regular embeddings

Proposition

Let N be a group of the same order as G. Suppose that

- N is an extension of A by B;
- $\delta : N \hookrightarrow \mathrm{Perm}(G)$ is a regular embedding;
- $J = \delta(A) \cdot 1_G$ is a normal subgroup of G.

Then

- $\delta_A : A \hookrightarrow \mathrm{Perm}(J)$ defined by $\delta_A(a) = \delta(a)$ is a regular embedding;
- $\delta_B : B \hookrightarrow \mathrm{Perm}(G/J)$ defined by $\delta_B(b)[gJ] = \delta(b)[gJ]$ is a regular embedding.

Extensions of regular embeddings

Proposition

Suppose that

- G is an extension of J by Q;
- A is a group with $|A| = |J|$, and $\alpha : A \hookrightarrow \mathrm{Perm}(J)$ is a regular embedding.
- B is a group with $|B| = |Q|$, and $\beta : B \hookrightarrow \mathrm{Perm}(Q)$ is a regular embedding.

Then, for each homomorphism $\varphi : B \to \mathrm{Aut}(A)$ and each 2-cocycle f for φ there is a distinct regular embedding $\delta : N(A, B, \varphi, f) \hookrightarrow \mathrm{Perm}(G)$ such that $\delta_A = \alpha$ and $\delta_B = \beta$.

Extensions of regular embeddings

Construction of δ.

- View N as $\{(a, b) \mid a \in A,\ b \in B\}$, and G as $\{[j, q] \mid j \in J,\ q \in Q\}$.
- Define $\delta : N \hookrightarrow \mathrm{Perm}(G)$ by setting

$$\delta(a, b)[1, 1] = [\alpha(a)[1], \beta(b)[1]],$$

  and insisting that δ be a homomorphism.
- Then

$$\begin{aligned} \delta(a, b)[j, q] &= \delta(a, b)\delta(a_j, b_q)[1, 1] \\ &= \delta((a, b)(a_q, b_q))[1, 1] \\ &= \delta(a(b \cdot a_q)f(b, b_q), bb_q)[1, 1] \\ &= [\alpha(a(b \cdot a_q)f(b, b_q))[1], \beta(bb_q)[1]]. \end{aligned}$$

Normalization for abelian extensions of classical structures

- Recall that J, Q are abelian groups.
- Let $G = G(J, Q, 1, g)$ and $N = (J, Q, 1, f)$.
- Let $A = J$ and $\alpha = \lambda : A \hookrightarrow \mathrm{Perm}(J)$.
- Let $B = Q$ and $\beta = \lambda : B \hookrightarrow \mathrm{Perm}(Q)$.
- Let $\delta : N \hookrightarrow \mathrm{Perm}(G)$ be constructed as above. Then

$$\delta(a, b)[1, 1] = [a, b]$$

and

$$\delta(a, b)[j, q] = [ajf(b, q), bq].$$

Proposition

$\delta(N)$ is normalized by $\lambda(G)$ if and only if $fg^{-1} : Q \times Q \to J$ is a bihomomorphism.

Normalization for abelian extensions of classical structures

Proof.

- Let $(a, b) \in N$ and $[\bar{j}, \overline{q}] \in G$.
- We need to show that there exists $(a', b') \in N$ such that

$$[\bar{j}, \overline{q}]\delta(a, b)[j, q] = \delta(a', b')[\bar{j}, \overline{q}][j, q] \tag{1}$$

  for all $[j, q] \in G$.
- Computing each side, we require

$$[a\bar{j}jf(b, q)g(\overline{q}, bq), b\overline{q}q] = [a'\bar{j}jf(b', \overline{q}q)g(\overline{q}, q), b'\overline{q}q] \tag{2}$$

  for all $[j, q] \in G$.
- Clearly it's necessary that $b' = b$.

Normalization for abelian extensions of classical structures

Continued...

- Setting $b' = b$ and studying the first components, we find we require (for all $[j, q] \in G$)

$$af(b,q)g(\overline{q},bq) = a'f(b',\overline{q}q)g(\overline{q},q)$$

That is,

$$\begin{aligned} a' &= a\frac{f(b,q)}{f(b,\overline{q}q)}\frac{g(\overline{q},bq)}{g(\overline{q},q)} \\ &= a\frac{f(\overline{q},q)}{g(\overline{q},q)}\frac{g(\overline{q},bq)}{f(\overline{q},bq)} \text{ by cocycle relation} \\ &= ah(\overline{q},q)h(\overline{q},bq)^{-1}, \text{ with } h = fg^{-1}. \end{aligned}$$

- We need $h(\overline{q},q)h(\overline{q},bq)^{-1}$ to be independent of q.

Normalization for abelian extensions of classical structures

Continued...

- We have set $h = fg^{-1}$, and we require $h(\overline{q}, q)h(\overline{q}, bq)^{-1}$ to be independent of q.
- This happens if and only if

$$h(\overline{q}, q)h(\overline{q}, bq)^{-1} = h(\overline{q}, b)^{-1} \text{ for all } q$$

that is, if and only if

$$h(\overline{q}, q)h(\overline{q}, b) = h(\overline{q}, bq) \text{ for all } q.$$

- Using the fact that all the groups involved are abelian, this is saying that h is a bihomomorphism.

□

The degree p^2 case

Theorem (Byott, 1999)

- Let p be a prime number and L/K a Galois extension of degree p^2 with group G. Choose $T \leq G$ of order p, and $d \in \{0, 1, \ldots, p-1\}$.
- Fix $\sigma, \tau \in G$ such that $\langle \tau \rangle = T$ and $\langle \sigma, \tau \rangle = G$.
- The regular subgroups of $\mathrm{Perm}(G)$ normalized by $\lambda(G)$ are the groups $N_{T,d} = \langle u, v \rangle$, with

$$\begin{aligned} u[\tau^l \sigma^k] &= \tau^{l+1} \sigma^k \\ v[\tau^l \sigma^k] &= \tau^{l+kd} \sigma^{k+1}, \end{aligned} \tag{3}$$

  We have $N_{T,d} \cong G$ unless $p = 2$ and $d = 1$.
- If $d = 0$ then $N_{T,d} = \lambda(G)$ regardless of T; the other choices give distinct subgroups. Thus L/K admits p Hopf-Galois structures if G is cyclic, and p^2 if G is elementary abelian.

The degree p^2 case

- How do these subgroups emerge from our approach?
- Fix a subgroup $J = \langle j \rangle$ of G of order p. Let $Q = \mathrm{Gal}(L^J/K) = \langle q \rangle$. Let $g : Q \times Q \to J$ be a 2-cocycle corresponding to G.
- Our construction says that $\delta(N)$ is regular and normalized by $\lambda(G)$ if and only if the 2-cocycle f corresponding to N satisfies $f = gh$ for some bihomomorphism $h : Q \times Q \to J$.

Proposition

The bihomomorphisms $Q \times Q \to J$ are precisely the maps

$$h_d(q^s, q^t) = j^{dst} \text{ for } d = 0, \ldots, p-1.$$

The degree p^2 case

- If G is elementary abelian then g is trivial, so $f = h_d$ for some d, and our construction yields

$$\begin{array}{rcl} \delta(j,1)[j^l, q^k] & = & [j^{l+1}, q^k] \\ \delta(1,q)[j^l, q^k] & = & [j^l j^{kd}, q^{k+1}] \end{array}$$

- If G is cyclic then things are slightly more complicated, but Byott's permutations do emerge.
- If p is odd then the bihomomorphisms are actually coboundaries, so $N \cong G$.
- If $p = 2$ then the nontrivial bihomomorphisms are not coboundaries, so $N \not\cong G$.

Generalizations

- What if we continue to assume that J, Q are abelian, but allow for nonabelian extensions of J by Q?

Theorem

- Let $G = G(J, Q, \theta, g)$, and write $q * j = \theta(q)[j]$.
- Let $N = N(J, Q, \varphi, f)$, and write $q \cdot j = \varphi(q)[j]$.
- Let $h = fg^{-1}$.

Then $\delta(N)$ is normalized by $\lambda(G)$ if and only if

- $q * (q' \cdot j) = q' \cdot (q * j)$ for all $q, q' \in Q$ and $j \in J$;
- $h(x, yz) = (y * h(x, z))(z \cdot h(x, y))$ for all $x, y, z \in Q$.

Problems

- Is all of this independent of the choices we could make at each point?
- There shouldn't be anything special about using $[1, 1]$ as the "basepoint" for the construction of δ. What is the effect of varying it?
- How many different embeddings do we get this way? How many different subgroups?
- The next simplest case to study is L/K of degree pq with p, q are distinct primes with $p \equiv 1 \pmod{q}$. Our construction does not produce all Hopf-Galois structures in this case. The definition of δ is too restrictive.

Thank you for your attention.