Hopf-Galois Structures and a Characterization of Dihedral Extensions

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1. Introduction

Assume $\mathbb{Q} \subseteq K$ and let L/K be a Galois extension with non-abelian group G.

Then L/K admits both a classical and canonical non-classical Hopf-Galois structure via the Hopf algebras $K[G]$ and H_λ, respectively.

By a theorem of C. Greither, $K[G] \cong H_\lambda$ as K-algebras.

In this talk we apply Greither's result to the case $K = \mathbb{Q}$, $G = D_3$ to yield a characterization of Galois extensions with group D_3.

In the case $G = D_4$, Greither's theorem has implications for a result of A. Ledet.

2. Hopf Galois theory

We recall the notion of a Hopf algebra, a Hopf-Galois extension, and the Greither-Pareigis classification.

A **bialgebra** over a field K is a K-algebra B together with K-algebra maps $\Delta : B \to B \otimes_K B$ (comultiplication) and $\varepsilon : B \to K$ (counit) which satisfy the conditions

$$(I \otimes \Delta)\Delta = (\Delta \otimes I)\Delta,$$

$$\mathrm{mult}(I \otimes \varepsilon)\Delta = I = \mathrm{mult}(\varepsilon \otimes I)\Delta,$$

where $\mathrm{mult} : B \otimes_K B \to B$ is the multiplication map of B and I is the identity map on B.

A **Hopf algebra** over K is a K-bialgebra H with a K-linear map $\sigma : H \to H$ which satisfies

$$\mathrm{mult}(I \otimes \sigma)\Delta(h) = \varepsilon(h)1_H = \mathrm{mult}(\sigma \otimes I)\Delta(h),$$

for all $h \in H$.

A K-Hopf algebra H is **cocommutative** if $\Delta = \tau \circ \Delta$, where $\tau : H \otimes_K H \to H \otimes_K H$, $a \otimes b \mapsto b \otimes a$ is the twist map.

Let L be a finite extension of K and let $\mathrm{m} : L \otimes_K L \to L$ denote multiplication in L.

Let H be a finite dimensional, cocommutative K-Hopf algebra and suppose there is a K-linear action of H on L which satisfies

$$h \cdot (xy) = (\mathrm{m} \circ \Delta)(h)(x \otimes y)$$
$$h \cdot 1 = \varepsilon(h)1$$

for all $h \in H,\ x, y \in L$, and that the K-linear map

$$j : L \otimes_K H \to \mathrm{End}_K(L),\ j(x \otimes h)(y) = x(h \cdot y)$$

is an isomorphism of vector spaces over K. Then we say H provides a **Hopf-Galois structure** on L/K.

Example 1. Suppose L/K is Galois with Galois group G. Let $H = K[G]$ be the group algebra, which is a Hopf algebra via $\Delta(g) = g \otimes g,\ \varepsilon(g) = 1$, $\sigma(g) = g^{-1}$, for all $g \in G$. The action

$$\left(\sum r_g g\right) \cdot x = \sum r_g(g(x))$$

provides the “usual” Hopf-Galois structure on L/K which we call the **classical** Hopf-Galois structure.

In general, the process of finding a Hopf algebra and constructing an action may seem daunting, but in the separable case C. Greither and B. Pareigis [4] have provided a complete classification of such structures.

Let L/K be separable with normal closure E. Let $G = \mathrm{Gal}(E/K)$, $G' = \mathrm{Gal}(E/L)$, and $X = G/G'$. Denote by $\mathrm{Perm}(X)$ the group of permutations of X.

A subgroup $N \leq \mathrm{Perm}(X)$ is **regular** if $|N| = |X|$ and $\eta[xG'] \neq xG'$ for all $\eta \neq 1_N, xG' \in X$.

Let $\lambda : G \to \mathrm{Perm}(X)$, $\lambda(g)(xG') = gxG'$, denote the left translation map. A subgroup $N \leq \mathrm{Perm}(X)$ is **normalized** by $\lambda(G) \leq \mathrm{Perm}(X)$ if $\lambda(G)$ is contained in the normalizer of N in $\mathrm{Perm}(X)$.

Theorem 2. (Greither-Pareigis) *Let L/K be a finite separable extension. There is a one-to-one correspondence between Hopf Galois structures on L/K and regular subgroups of $\mathrm{Perm}(X)$ that are normalized by $\lambda(G)$.*

One direction of this correspondence works by Galois descent: Let N be a regular subgroup normalized by $\lambda(G)$. Then G acts on the group algebra $E[N]$ through the Galois action on E and conjugation by $\lambda(G)$ on N, i.e.,

$$g(x\eta) = g(x)(\lambda(g)\eta\lambda(g^{-1})), g \in G,\ x \in E,\ \eta \in N.$$

For simplicity, we will denote the conjugation action of $\lambda(g) \in \lambda(G)$ on $\eta \in N$ by ${}^g\eta$.

We then define

$$H = (E[N])^G = \{x \in E[N] :\ g(x) = x, \forall g \in G\}.$$

The action of H on L/K is thus

$$\left(\sum_{\eta\in N} r_\eta \eta\right)\cdot x = \sum_{\eta\in N} r_\eta \eta^{-1}[1_G](x),$$

see [2, Proposition 1].

The fixed ring H is an n-dimensional K-Hopf algebra, $n = [L : K]$, and L/K has a Hopf Galois structure via H [4, p. 248, proof of 3.1 (b)$\Longrightarrow$ (a)], [1, Theorem 6.8, pp. 52-54].

By [4, p. 249, proof of 3.1, (a) $\Longrightarrow$ (b)],

$$E \otimes_K H \cong E \otimes_K K[N] \cong E[N],$$

as E-Hopf algebras, that is, H is an E-**form** of $K[N]$.

Theorem 2 can be applied to the case where L/K is Galois with group G (thus, $E = L$, $G' = 1_G$, $G/G' = G$). In this case the Hopf Galois structures on L/K correspond to regular subgroups of $\mathrm{Perm}(G)$ normalized by $\lambda(G)$, where $\lambda : G \to \mathrm{Perm}(G)$, $\lambda(g)(h) = gh$, is the left regular representation.

Example 3. Suppose L/K is a Galois extension, $G = \mathrm{Gal}(L/K)$. Let $\rho : G \to \mathrm{Perm}(G)$ be the right regular representation defined as $\rho(g)(h) = hg^{-1}$ for $g, h \in G$. Then $\rho(G)$ is a regular subgroup normalized by $\lambda(G)$, since $\lambda(g)\rho(h)\lambda(g^{-1}) = \rho(h)$ for all $g, h \in G$; N corresponds to a Hopf-Galois structure with K-Hopf algebra $H = L[\rho(G)]^G = K[G]$, the usual group ring Hopf algebra with its usual action on L. Consequently, $\rho(G)$ corresponds to the **classical** Hopf Galois structure.

Example 4. Again, suppose L/K is Galois with group G. Let $N = \lambda(G)$. Then N is a regular subgroup of $\mathrm{Perm}(G)$ which is normalized by $\lambda(G)$, and $N = \rho(G)$ if and only if N abelian. We denote the corresponding Hopf algebra by H_λ. If G is non-abelian, then $\lambda(G)$ corresponds to the **canonical non-classical** Hopf-Galois structure.

3. Isomorphism Classes

It is of interest to determine how $K[G]$ and H_λ fall into K-Hopf algebra and K-algebra isomorphism classes. We have:

Theorem 5. (Koch, Kohl, Truman, U.) *Assume that G is non-abelian. Then $H_\lambda \ncong K[G]$ as K-Hopf algebras.*

Proof. Over L, $K[G]$ and H_λ are isomorphic to $L[G]$ as Hopf algebras, thus their duals $K[G]^*$ and H_λ^* are finite dimensional as algebras over K and separable (as defined in [8, 6.4, page 47]). Using the classification of such K-algebras [8, 6.4, Theorem], we conclude that $K[G]^*$ and H_λ^* are not isomorphic as K-Hopf algebras, and so neither are $K[G]$ and H_λ. In fact, by [8, 6.3, Theorem], $K[G]^*$ and H_λ^* are not isomorphic as K-algebras, and consequently, $K[G]$ and H_λ are not isomorphic as K-coalgebras. $\square$

On the other hand, C. Greither has shown that following.

Theorem 6. (Greither) *$H_\lambda \cong K[G]$ as K-algebras.*

Proof. (Sketch.)

Step 1. Obtain the Wedderburn-Artin decomposition of $K[G]$, thus:

$$K[G] \cong A_1 \times A_2 \times \cdots \times A_m,$$

where $A_i = \mathrm{Mat}_{n_i}(E_i)$ for division rings E_i.

Step 2. Show that the action of G on $L[G]$ restricts to an action on the components $L \otimes A_i$ of $L[G] \cong L \otimes_K K[G]$, and hence each component $L \otimes A_i$ descends to a component S_i in the Wedderburn-Artin decomposition of H_λ; (supressing subscripts) S is an L-form of A.

Step 3. L-forms of A are classified by the pointed set $H^1(G, \mathrm{Aut}(L \otimes_K A))$. Let $[\hat{f}]$ be the class corresponding to the class of S.

Step 4. There exists a map in cohomology

$$\Psi : H^1(G, GL_n(L \otimes_K E)) \to H^1(G, \mathrm{Inn}(L \otimes_K A))$$

with $[\hat{f}] \in H^1(G, \mathrm{Inn}(L \otimes_K A))$. Moreover, there exists a class $[\hat{q}] \in H^1(G, GL_n(L \otimes_K E))$ with $\Psi([\hat{q}]) = [\hat{f}]$.

Step 5. By Hilbert's Theorem 90 (or its generalization) $H^1(G, GL_n(L \otimes_K E))$ is trivial, hence $[\hat{f}]$ is trivial, so $S \cong A$ as K-algebras, thus $H_\lambda \cong K[G]$ as K-algebras. $\square$

4. Dihedral Extensions

Let D_n denote the dihedral group of order $2n$ for $n \geq 3$. Explicitly, we write

$$D_n = \langle \sigma, \tau : \sigma^n = \tau^2 = \sigma\tau\sigma\tau = 1 \rangle.$$

Let L/K be a Galois extension with group D_n.
By Example 3 and Example 4 we have regular subgroups $\rho(D_n),\ \lambda(D_n)$ normalized by $\lambda(D_n)$.

These regular subgroups give rise to the classical and canonical non-classical Hopf-Galois structures on L/K via the K-Hopf algebras $K[D_n]$ and H_λ, respectively.

Example 7. In the case L/K is Galois with group D_n, the classical Hopf-Galois structure on L/K has K-Hopf algebra

$$K[D_n] = \left\{ \sum_{i=0}^{n-1} \sum_{j=0}^{1} a_{i,j}\sigma^i\tau^j : \ a_{i,j} \in K \right\}.$$

Example 8. In the case L/K is Galois with group D_3, then by [1, Example 6.12], the canonical non-classical Hopf-Galois structure on L/K has K-Hopf algebra

$$H_\lambda = \{a_0 + a_1\sigma + \tau(a_1)\sigma^2 + b_0\tau + \sigma(b_0)\tau\sigma + \sigma^2(b_0)\tau\sigma^2 :$$

$$a_0 \in \mathbb{Q}, a_1 \in L^{\langle\sigma\rangle}, b_0 \in L^{\langle\tau\rangle}\}.$$

Lemma 9. *Let $L/\mathbb{Q}$ be a Galois extension with group D_4. Then H_λ consists of elements of the form*

$$h = a_0 + a_1\sigma + a_2\sigma^2 + \tau(a_1)\sigma^3 + b_0\tau + b_1\tau\sigma + \sigma(b_0)\tau\sigma^2 + \sigma(b_1)\tau\sigma^3,$$

where $a_0, a_2 \in \mathbb{Q}$, $a_1 \in L^{\langle\sigma\rangle}$, $b_0 \in L^{\langle\sigma^2,\tau\rangle}$, and $b_1 \in L^{\langle\sigma^2,\tau\sigma^3\rangle}$.

Proof. Following [1, Example 6.12], let

$$x = a_0 + a_1\sigma + a_2\sigma^2 + a_3\sigma^3 + b_0\tau + b_1\tau\sigma + b_2\tau\sigma^2 + b_3\tau\sigma^3$$

be an element of LD_4 for some $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3 \in L$. Then the elements in H_λ are precisely those x for which $\tau(x) = x$ and $\sigma(x) = x$. □

5. Application to D_3

Let $L/\mathbb{Q}$ be a Galois extension with group D_3. Necessarily, $L = \mathbb{Q}(\alpha, \sqrt{\mathcal{D}})$, where α is a root of a reduced irreducible cubic $p(x) = x^3 + bx - c$ over $\mathbb{Q}$, and $\mathcal{D} = -4b^3 - 27c^2$ is the discriminant of $p(x)$. Note that $\mathcal{D}$ is not a square in $\mathbb{Q}$.

We have two Hopf-Galois structures on $L/\mathbb{Q}$, one is the classical Hopf-Galois structure via the $\mathbb{Q}$-Hopf algebra $\mathbb{Q}[D_3]$, and the other is the canonical Hopf-Galois structure via the $\mathbb{Q}$-Hopf algebra H_λ.

By Theorem 6, $H_\lambda \cong \mathbb{Q}[D_3]$ as $\mathbb{Q}$-algebras.

And by a well-known result, the Wedderburn-Artin decomposition of $\mathbb{Q}[D_3]$ is

$$\mathbb{Q} \times \mathbb{Q} \times \mathrm{Mat}_2(\mathbb{Q}).$$

Thus, the decomposition of H_λ is

$$\mathbb{Q} \times \mathbb{Q} \times \mathrm{Mat}_2(\mathbb{Q}).$$

So, H_λ contains a non-trivial nilpotent element h of index 2 (corresponding to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the component $\mathrm{Mat}_2(\mathbb{Q})$.)

We have:

$$h^2 = 0, \quad h \neq 0.$$

But as we have seen in Example 8 above, this element must be of the form

$$h = a_0 + a_1\sigma + \tau(a_1)\sigma^2 + b_0\tau + \sigma(b_0)\tau\sigma + \sigma^2(b_0)\tau\sigma^2,$$

for some $a_0 \in \mathbb{Q}$, $a_1 \in L^{\langle\sigma\rangle}$, $b_0 \in L^{\langle\tau\rangle}$.

From this we obtain:

Theorem 10.(Koch, Kohl, Truman, U.) *Let $L/\mathbb{Q}$ be a Galois extension with group D_3. Then L is the splitting field of an irreducible cubic $x^3 + bx - c$ where $-b\mathcal{D}$ is a square in $\mathbb{Q}$.*

Proof. As we have seen above, H_λ contains a non-trivial element h with $h^2 = 0$. By direct computation
$h^2 =$

$$
\begin{aligned}
& a_0^2 + a_0 a_1 \sigma + a_0 \tau(a_1)\sigma^2 + a_0 b_0 \tau + a_0 \sigma(b_0)\tau\sigma + a_0 \sigma^2(b_0)\tau\sigma^2 \\
& + a_0 a_1 \sigma + a_1^2 \sigma^2 + a_1 \tau(a_1) + a_1 b_0 \tau\sigma^2 + a_1 \sigma(b_0)\tau + a_1 \sigma^2(b_0)\tau\sigma \\
& + a_0 \tau(a_1)\sigma^2 + a_1 \tau(a_1) + \tau(a_1^2)\sigma + b_0 \tau(a_1)\tau\sigma + \tau(a_1)\sigma(b_0)\tau\sigma^2 \\
& \qquad + \tau(a_1)\sigma^2(b_0)\tau \\
& + a_0 b_0 \tau + a_1 b_0 \tau\sigma + b_0 \tau(a_1)\tau\sigma^2 + b_0^2 + b_0 \sigma(b_0)\sigma + b_0 \sigma^2(b_0)\sigma^2 \\
& + a_0 \sigma(b_0)\tau\sigma + a_1 \sigma(b_0)\tau\sigma^2 + \sigma(b_0)\tau(a_1)\tau + b_0 \sigma(b_0)\sigma^2 + \sigma(b_0^2) \\
& \qquad + \sigma(b_0)\sigma^2(b_0)\sigma \\
& + a_0 \sigma^2(b_0)\tau\sigma^2 + a_1 \sigma^2(b_0)\tau + \tau(a_1)\sigma^2(b_0)\tau\sigma + b_0 \sigma^2(b_0)\sigma \\
& \qquad + \sigma(b_0)\sigma^2(b_0)\sigma^2 + \sigma^2(b_0^2).
\end{aligned}
$$

Hence,

$$h^2 = Z_1 + Z_\sigma \sigma + Z_{\sigma^2}\sigma^2 + Z_\tau \tau + Z_{\tau\sigma}\tau\sigma + Z_{\tau\sigma^2}\tau\sigma^2 = 0,$$

where

$$\begin{aligned}
Z_1 &= a_0^2 + 2a_1\tau(a_1) + b_0^2 + \sigma(b_0^2) + \sigma^2(b_0^2) \\
Z_\sigma &= 2a_0a_1 + \tau(a_1^2) + b_0\sigma(b_0) + \sigma(b_0)\sigma^2(b_0) + b_0\sigma^2(b_0) \\
Z_{\sigma^2} &= 2a_0\tau(a_1) + a_1^2 + b_0\sigma(b_0) + \sigma(b_0)\sigma^2(b_0) + b_0\sigma^2(b_0) \\
Z_\tau &= 2a_0b_0 + (a_1 + \tau(a_1))\sigma(b_0) + (a_1 + \tau(a_1))\sigma^2(b_0) \\
Z_{\tau\sigma} &= 2a_0\sigma(b_0) + (a_1 + \tau(a_1))b_0 + (a_1 + \tau(a_1))\sigma^2(b_0) \\
Z_{\tau\sigma^2} &= 2a_0\sigma^2(b_0) + (a_1 + \tau(a_1))b_0 + (a_1 + \tau(a_1))\sigma(b_0).
\end{aligned}$$

Thus

$$Z_1 = Z_\sigma = Z_{\sigma^2} = Z_\tau = Z_{\tau\sigma} = Z_{\tau\sigma^2} = 0,$$

and from this system, the result follows. □

Example 11. Let L be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Then $L/\mathbb{Q}$ is Galois with group D_3. Here, $\mathcal{D} = -108$ which is not a square in $\mathbb{Q}$. However, $-b\mathcal{D} = 0 \cdot -108 = 0$ is a square in $\mathbb{Q}$.

H_λ contains the non-trivial nilpotent element of index 2:

$$h = \sqrt[3]{2}\tau + \sqrt[3]{2}\zeta_3\tau\sigma + \sqrt[3]{2}\zeta_3^2\tau\sigma^2.$$

Example 12. Let L be the splitting field of $p(x) = x^3 + 23x - 529$ over $\mathbb{Q}$. As one can check, $p(x)$ is irreducible over $\mathbb{Q}$, and $\mathcal{D} = -7604375$ is not a square in $\mathbb{Q}$. Hence $L/\mathbb{Q}$ is Galois with group D_3. Now

$$-b\mathcal{D} = 174900625 = 13225^2.$$

The splitting field of $p(x)$ is $L = \mathbb{Q}(b_0, \sqrt{-23})$, where b_0 is a root of $p(x)$. Moreover, H_λ contains the non-trivial nilpotent index 2 element

$$h = \sqrt{-23}\sigma - \sqrt{-23}\sigma^2 + b_0\tau + \sigma(b_0)\tau\sigma + \sigma^2(b_0)\tau\sigma^2.$$

Example 13. Let $p(x) = x^3 - 4x + 1$. Then $p(x)$ is irreducible with $\mathcal{D} = 229$ and so the splitting field of $p(x)$ over $\mathbb{Q}$ is Galois with group D_3. However, $-b\mathcal{D} = 4 \cdot 229$, which is not a square in $\mathbb{Q}$.

By Theorem 6,

$$H_\lambda \cong \mathbb{Q} \times \mathbb{Q} \times \mathrm{Mat}_2(\mathbb{Q}),$$

and hence H_λ contains a non-trivial nilpotent element h with $h^2 = 0$.

Theorem 10 tells us how to construct from this h an irreducible cubic $x^3 + b'x - c'$ with discriminant $\mathcal{D}'$ whose splitting field is the same as that of $p(x)$, and which satisfies $-b'\mathcal{D}'$ a square in $\mathbb{Q}$.

6. Application to D_4

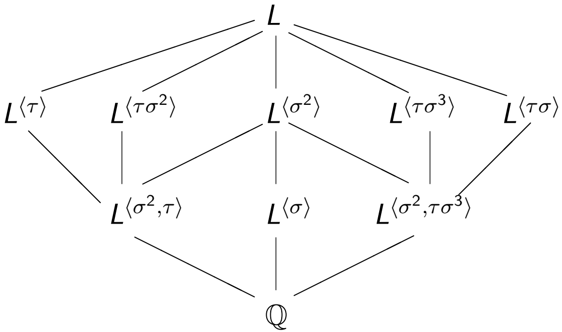
Let $L/\mathbb{Q}$ be Galois with group D_4. By (Curtis and Reiner)

$$\mathbb{Q}[D_4] \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathrm{Mat}_2(\mathbb{Q}),$$

and so, by Theorem 6,

$$H_\lambda \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathrm{Mat}_2(\mathbb{Q}).$$

The lattice of fixed fields is:



Note that $L^{\langle \sigma^2 \rangle}$ is the unique biquadratic extension of $\mathbb{Q}$ contained in L.

We have $L^{\langle \sigma^2 \rangle} = \mathbb{Q}(\alpha, \beta)$ with $L^{\langle \sigma^2, \tau \rangle} = \mathbb{Q}(\beta)$, $L^{\langle \sigma \rangle} = \mathbb{Q}(\alpha)$ and $L^{\langle \sigma^2, \tau\sigma^3 \rangle} = \mathbb{Q}(\alpha\beta)$.

Thus $b_0 = b_{0,1} + b_{0,2}\beta$, $a_1 = a_{1,1} + a_{1,2}\alpha$, and $b_1 = b_{1,1} + b_{1,2}\alpha\beta$ for some $b_{0,1}, b_{0,2}, a_{1,1}, a_{1,2}, b_{1,1}, b_{1,2} \in \mathbb{Q}$.

We have $\sigma(b_0) = b_{0,1} - b_{0,2}\beta$, $\sigma(b_1) = b_{1,1} - b_{1,2}\alpha\beta$, and $\tau(a_1) = a_{1,1} - a_{1,2}\alpha$.

Lemma 14. *The component* $\mathrm{Mat}_2(\mathbb{Q})$ *in the decomposition of* H_λ *has* $\mathbb{Q}$*-basis*

$$\{(1-\sigma^2)/2, \alpha(\sigma-\sigma^3), \beta(\tau-\tau\sigma^2), \alpha\beta(\tau\sigma-\tau\sigma^3)\}.$$

Proof. The idempotents corresponding to the 4 copies of $\mathbb{Q}$ in the decomposition of H_λ are $e_i = \frac{1}{8}\sum_{s\in D_4}\chi_i(s^{-1})s$, $1 \le i \le 4$, where χ_i are the characters of the 4 1-dimensional irreducible representations of D_4 (each e_i is in LD_4 and is fixed by D_4, hence $e_i \in H_{\lambda,4}$).

The idempotent corresponding to the component $\mathrm{Mat}_2(\mathbb{Q})$ is

$$e = 1 - \sum_{i=1}^{4} e_i = \frac{1-\sigma^2}{2}.$$

By Lemma 9, a typical element of H_λ appears as

$$h = a_0 + a_1\sigma + a_2\sigma^2 + \tau(a_1)\sigma^3 + b_0\tau + b_1\tau\sigma + \sigma(b_0)\tau\sigma^2 + \sigma(b_1)\tau\sigma^3,$$

where $a_0, a_2 \in \mathbb{Q}$, $a_1 \in L^{\langle\sigma\rangle}$, $b_0 \in L^{\langle\sigma^2,\tau\rangle}$, and $b_1 \in L^{\langle\sigma^2,\tau\sigma^3\rangle}$.

Thus a typical element of $\mathrm{Mat}_2(\mathbb{Q})$ is

$$\begin{aligned} eh &= \left(\frac{1-\sigma^2}{2}\right)\big(a_0 + a_1\sigma + a_2\sigma^2 + \tau(a_1)\sigma^3 + b_0\tau + b_1\tau\sigma \\ &\quad + \sigma(b_0)\tau\sigma^2 + \sigma(b_1)\tau\sigma^3\big) \\ &= q\left(\frac{1-\sigma^2}{2}\right) + a_{1,2}\alpha(\sigma-\sigma^3) + b_{0,2}\beta(\tau-\tau\sigma^2) \\ &\quad + b_{1,2}\alpha\beta(\tau\sigma-\tau\sigma^3), \end{aligned}$$

for $q, a_{1,2}, b_{0,2}, b_{1,2} \in \mathbb{Q}$. Thus

$$\{(1-\sigma^2)/2, \alpha(\sigma-\sigma^3), \beta(\tau-\tau\sigma^2), \alpha\beta(\tau\sigma-\tau\sigma^3)\}$$

is a $\mathbb{Q}$-basis for $\mathrm{Mat}_2(\mathbb{Q})$. $\square$

Theorem 15. *Let $L/\mathbb{Q}$ be a Galois extension with group D_4. Then there exists a non-trivial solution (b, c, d) in $\mathbb{Q}$ of the equation*

$$b^2\alpha^2 = c^2\beta^2 + d^2\alpha^2\beta^2, \tag{1}$$

where $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$ is the unique biquadratic extension contained in L, with $\alpha^2, \beta^2 \in \mathbb{Q}$.

Proof. By Lemma 14, the component $\mathrm{Mat}_2(\mathbb{Q})$ has $\mathbb{Q}$-basis

$$\{(1-\sigma^2)/2, \alpha(\sigma-\sigma^3), \beta(\tau-\tau\sigma^2), \alpha\beta(\tau\sigma-\tau\sigma^3)\}.$$

Put $1 := (1-\sigma^2)/2$, $X := \alpha(\sigma-\sigma^3)$, $Y := \beta(\tau-\tau\sigma^2)$, and $Z := \alpha\beta(\tau\sigma-\tau\sigma^3)$.

Then we have the multiplication table:

	1	X	Y	Z
1	1	X	Y	Z
X	X	$-4\alpha^2$	$-2Z$	$2\alpha^2 Y$
Y	Y	$2Z$	$4\beta^2$	$2\beta^2 X$
Z	Z	$-2\alpha^2 Y$	$-2\beta^2 X$	$4\alpha^2\beta^2$

Clearly, $\mathrm{Mat}_2(\mathbb{Q}) \subseteq H_\lambda$ contains an element $h \in$ with $h^2 = 0$ and $h \neq 0$. Write

$$h = a + bX + cY + dZ,$$

for $a, b, c, d \in \mathbb{Q}$. Then

$$\begin{aligned} h^2 &= (a + bX + cY + dZ)(a + bX + cY + dZ) \\ &= (a^2 - 4b^2\alpha^2 + 4c^2\beta^2 + 4d^2\alpha^2\beta^2) + 2abX + 2acY + 2adZ \\ &= 0, \end{aligned}$$

and so,

$$
\begin{aligned}
a^2 + 4c^2\beta^2 + 4d^2\alpha^2\beta^2 &= 4b^2\alpha^2 \\
2ab &= 0 \\
2ac &= 0 \\
2ad &= 0.
\end{aligned}
$$

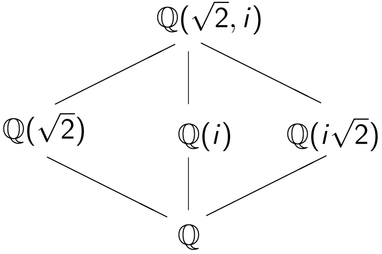
If $a \neq 0$, then $b = c = d = 0$, hence $a^2 = 0$, which is impossible. So we assume that $a = 0$.

It follows that $h^2 = 0$, $h \neq 0$, implies that there is a non-trivial solution (b, c, d) to

$$b^2\alpha^2 = c^2\beta^2 + d^2\alpha^2\beta^2.$$

Moreover, h is non-trivial if and only if (b, c, d) is non-trivial. $\square$

Example 16. Let L be the splitting field of $x^4 - 2$ over $\mathbb{Q}$. By [3, Corollary 4.5], the Galois group is D_4. We have $L = \mathbb{Q}(\sqrt[4]{2}, i)$ with $\sigma(i) = i$, $\sigma(\sqrt[4]{2}) = -i\sqrt[4]{2}$, $\tau(i) = -i$, and $\tau(\sqrt[4]{2}) = \sqrt[4]{2}$. The lattice of the unique biquadratic extension is



where $L^{\langle\sigma^2\rangle} = \mathbb{Q}(\sqrt{2}, i)$ is the unique biquadratic extension in L with quadratic subfields $L^{\langle\sigma^2,\tau\rangle} = \mathbb{Q}(\sqrt{2})$, $L^{\langle\sigma\rangle} = \mathbb{Q}(i)$, and $L^{\langle\sigma^2,\tau\sigma^3\rangle} = \mathbb{Q}(i\sqrt{2})$. We choose $\beta = \sqrt{2}$, $\alpha = i$. Then equation (1) is

$$-b^2 = 2c^2 - 2d^2,$$

which has a non-trivial solution $(b, c, d) = (0, 1, 1)$. The corresponding element $h \in \mathrm{Mat}_2(\mathbb{Q})$ is

$$h = \sqrt{2}(\tau - \tau\sigma^2) + i\sqrt{2}(\tau\sigma - \tau\sigma^3),$$

which satisfies $h^2 = 0$, $h \neq 0$.

Example 17. Let $f(x) = x^4 - 4x^2 - 3$. Then $p(x) = f(x-1)$ is irreducible over $\mathbb{Q}$, by the Eisenstein criterion, and hence $f(x)$ is irreducible over $\mathbb{Q}$. By [3, Corollary 4.5] the Galois group of the splitting field of $f(x)$ is D_4. Note that the discriminant satisfies

$$\mathcal{D} = -37632 = -3 \cdot 12544 = -3 \cdot 112^2 = -147 \cdot 16^2.$$

The roots of $f(x)$ are

$$\sqrt{2+\sqrt{7}},\ \sqrt{2-\sqrt{7}},\ -\sqrt{2+\sqrt{7}},\ -\sqrt{2-\sqrt{7}}.$$

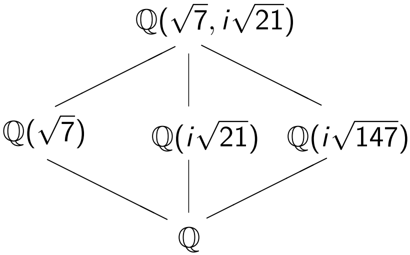
The splitting field over $\mathbb{Q}$ is $L = \mathbb{Q}(\sqrt{2+\sqrt{7}}, i\sqrt{3})$. The Galois action is given by

$$\sigma\left(\sqrt{2+\sqrt{7}}\right) = \sqrt{2-\sqrt{7}},\ \ \sigma\left(\sqrt{2-\sqrt{7}}\right) = -\sqrt{2+\sqrt{7}},$$

$$\tau\left(\sqrt{2+\sqrt{7}}\right) = \sqrt{2+\sqrt{7}},\ \ \tau\left(\sqrt{2-\sqrt{7}}\right) = -\sqrt{2-\sqrt{7}},$$

$$\sigma(i\sqrt{3}) = \tau(i\sqrt{3}) = -i\sqrt{3}.$$

The unique biquadratic extension contained in L is $\mathbb{Q}(\sqrt{7}, i\sqrt{3}) = \mathbb{Q}(\sqrt{7}, i\sqrt{21})$, with lattice



where $L^{\langle\sigma^2\rangle} = \mathbb{Q}(\sqrt{7}, i\sqrt{21})$, with quadratic subfields $L^{\langle\sigma^2,\tau\rangle} = \mathbb{Q}(\sqrt{7})$, $L^{\langle\sigma\rangle} = \mathbb{Q}(i\sqrt{21})$, and $L^{\langle\sigma^2,\tau\sigma^3\rangle} = \mathbb{Q}(i\sqrt{37632}) = \mathbb{Q}(i\sqrt{147}) = \mathbb{Q}(i\sqrt{3})$, see [3, proof of Theorem 4.1]. Let $\beta = \sqrt{7}$, $\alpha = i\sqrt{21}$. Then equation (1) is

$$-21b^2 = 7c^2 - 147d^2,$$

which has non-trivial solution $(b, c, d) = (1, 9, 2)$. Thus

$$i\sqrt{21}(\sigma - \sigma^3) + 9\sqrt{7}(\tau - \tau\sigma^2) + 2i\sqrt{147}(\tau\sigma - \tau\sigma^3)$$

is a non-trivial nilpotent element of index 2 in H_λ.

7. Application to a Result of Ledet

We obtain a new proof of the following result of A. Ledet [6, 0.4]:

Theorem 18.(Ledet) *Let $L/\mathbb{Q}$ be a Galois extension with group D_4. Let $\mathbb{Q}(\alpha,\beta)/\mathbb{Q}$ be the unique biquadratic extension contained in L. Then $\beta^2\alpha^2$ is a norm in $\mathbb{Q}(\beta)/\mathbb{Q}$.*

Proof. By Theorem 15 there exists a non-trivial solution (b, c, d) in $\mathbb{Q}$ to the equation

$$b^2\alpha^2 = c^2\beta^2 + d^2\alpha^2\beta^2.$$

Assuming $c \neq 0$, $\alpha \neq 0$, we have

$$\frac{b^2}{c^2} - \frac{d^2}{c^2}\beta^2 = \frac{\beta^2}{\alpha^2},$$

thus $\frac{\beta^2}{\alpha^2}$ is a norm in $\mathbb{Q}(\beta)/\mathbb{Q}$. Consequently, $\beta^2\alpha^2$ is a norm in $\mathbb{Q}(\beta)/\mathbb{Q}$. □

Ledet's result also gives another proof of Greither's result (Theorem 6) in the case $G = D_4$:

Theorem 19. *Let $L/\mathbb{Q}$ be a Galois extension with group D_4. Then*

$$H_\lambda \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathrm{Mat}_2(\mathbb{Q}).$$

Proof. Regardless of Greither's result, we always have

$$H_\lambda \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathrm{Mat}_r(R),$$

where $1 \leq r \leq 2$ and R is some division ring.

Now, $L/\mathbb{Q}$ is a solution to the "Galois theoretical embedding problem" given by $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$ and the short exact sequence

$$1 \to \langle \sigma^2 \rangle \to D_4 \to C_2 \times C_2 \to 1.$$

So by [6, 0.4], $\beta^2\alpha^2$ is a norm in $\mathbb{Q}(\beta)/\mathbb{Q}$, that is, there exist $x, y \in \mathbb{Q}$ so that

$$x^2 - y^2\beta^2 = \beta^2\alpha^2.$$

Thus,

$$x^2 = \beta^2\alpha^2 + y^2\beta^2, \quad \text{or}$$
$$x^2\alpha^2 = \alpha^4\beta^2 + y^2\alpha^2\beta^2.$$

Let $b = x$, $c = \alpha^2$, $d = y$. Then

$$bX + cY + dZ$$

is a non-trivial nilpotent of index 2 in H_λ, thus

$$H_\lambda \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathrm{Mat}_2(\mathbb{Q}).$$

□

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