Hopf orders in $(KC_p^3)^*$ over a discrete valuation ring of characteristic p

Robert G. Underwood
Department of Mathematics and Computer Science
Auburn University at Montgomery
Montgomery, Alabama

AUBURN
MONTGOMERY

March 2, 2019

1. Introduction

Let p be prime, let R be a discrete valuation ring of characteristic p and quotient field K, with uniformizing parameter π and valuation $\nu_K : K \to \mathbb{Z}$. Let C_p^n denote the elementary abelian group of order p^n. Let KC_p^n be the group ring Hopf algebra with dual Hopf algebra $(KC_p^n)^*$.

This talk concerns the structure of R-Hopf orders in $(KC_p^n)^*$ for $n \geq 1$. The cases $n = 1, 2$ are known; complete classifications have been given by J. Tate and F. Oort in the case $n = 1$, and G. Elder and U. in the case $n = 2$. For $n = 1$, one parameter is required to determine the Hopf order, and for $n = 2$ we require three parameters.

For arbitrary n, A. Koch has recently shown that Hopf orders in $(KC_p^n)^*$ are completely classified using $n(n+1)/2$ parameters.

What remains unsettled is the explicit structure of the Hopf orders in $(KC_p^n)^*$ (and their duals in KC_p^n).

Towards this end, we determine the algebraic structure of all Hopf orders in $(KC_p^3)^*$ and conjecture about the structure of their duals in KC_p^3.

We begin with a review of the $n = 1, 2$ cases.

2. Hopf orders in $(KC_p)^*$

Let σ generate C_p. Then it is well-known that the group ring KC_p is a K-Hopf algebra. Let $i \geq 0$ be an integer and let

$$\mathcal{H}_i = R\left[\frac{\sigma - 1}{\pi^i}\right].$$

Since $(\sigma - 1)^p = 0$ in KC_p, it is easy to see that $\mathcal{H}_i$ is both closed under multiplication and a free R-module of rank p. Since $RC_p \subseteq \mathcal{H}_i$, we clearly have $K\mathcal{H}_i = KC_p$.

Comultiplication on σ is grouplike, therefore, letting $x = (\sigma - 1)/\pi^i$ we have

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \pi^i x \otimes x \in \mathcal{H}_i \otimes \mathcal{H}_i.$$

As a result, $\mathcal{H}_i$ is a Hopf order in KC_p.

Let $(KC_p)^*$ be the linear dual of KC_p, and let $\{e_i\}_{i\in\mathbb{F}_p}$ be the K-basis for KC_p^* which is dual to the basis $\{\sigma^j\}_{j\in\mathbb{F}_p}$ for KC_p. We have $\langle e_i, \sigma^j\rangle = \delta_{i,j}$, the Kronecker delta function.

It is well-known that $(KC_p)^*$ is a K-Hopf algebra. Multiplication in $(KC_p)^*$ is determined by $e_i e_j = \delta_{i,j}$. Thus $\{e_i\}_{i\in\mathbb{F}_p}$ is an orthonormal basis, and $e_0 + e_1 + \cdots + e_{p-1}$ is the multiplicative identity. The counit is determined by $\varepsilon(e_i) = \delta_{i,0}$, comultiplication is determined by $\Delta(e_i) = \sum_{j\in\mathbb{F}_p} e_j \otimes e_{i-j}$, and the antipode satisfies $S(e_i) = e_{-i}$.

Lemma 2.1. *Let* $\xi_1 = \sum_{r=1}^{p-1} re_r \in (KC_p)^*$*. Then* $\langle \xi_1, (\sigma-1)^j \rangle = \delta_{1,j}$ *and* $(RC_p)^*$ *is an R-Hopf algebra with* $(RC_p)^* = R[\xi_1]$ *where* $\xi_1^p = \xi_1$*. The counit map satisfies* $\varepsilon(\xi_1) = 0$*, comultiplication is given as* $\Delta(\xi_1) = \xi_1 \otimes 1 + 1 \otimes \xi_1$*, namely* ξ_1 *is primitive, and the antipode satisfies* $S(\xi_1) = -\xi_1$*.*

Proposition 2.2. *Let* $i \geq 0$ *be an integer and let* $\beta = \pi^i \xi_1$*. Then* $R[\beta]$ *is an R-Hopf algebra contained in* $(RC_p)^*$ *with* $\beta^p = \pi^{(p-1)i}\beta$*; its coalgebra structure is defined by counit* $\varepsilon(\beta) = 0$*, comultiplication* $\Delta(\beta) = \beta \otimes 1 + 1 \otimes \beta$*, and antipode* $S(\beta) = -\beta$*. We have* $R[\beta] = \mathcal{H}_i^*$*.*

Theorem 2.3. [Tate-Oort] *Every Hopf order in* $(KC_p)^*$ *can be written as* $R[\beta] = R[\pi^i \xi_i]$ *for some* $i \geq 0$*.*

Corollary 2.4. *Every Hopf order in* KC_p *can be written as* $\mathcal{H}_i$ *for some* $i \geq 0$*.*

3. Hopf orders in $(KC_p^2)^*$

Let $C_p^2 = \langle \sigma_1, \sigma_2 \rangle$. Then $\{\sigma_1^a \sigma_2^b\}_{a,b \in \mathbb{F}_p}$ is a basis for KC_p^2, with dual basis $\{e_{a,b}\}_{a,b \in \mathbb{F}_p}$ for $(KC_p^2)^*$ satisfying $\langle e_{a,b}, \sigma_1^c \sigma_2^d \rangle = \delta_{a,c}\delta_{b,d}$.

The dual $(KC_p^2)^*$ is a K-Hopf algebra. Multiplication in $(KC_p^2)^*$ is given by $e_{a,b}e_{c,d} = \delta_{a,c}\delta_{b,d}e_{c,d}$, hence $\{e_{a,b}\}_{a,b \in \mathbb{F}_p}$ is an orthonormal basis with $\sum_{a,b \in \mathbb{F}_p} e_{a,b} = 1 \in (KC_p^2)^*$.

The counit map is determined by $\varepsilon(e_{a,b}) = \delta_{a,0}\delta_{b,0}$,
comultiplication is determined by $\Delta(e_{a,b}) = \sum_{i,j \in \mathbb{F}_p} e_{i,j} \otimes e_{a-i,b-j}$,
and the antipode satisfies $S(e_{a,b}) = e_{-a,-b}$.

We identify $(KC_p^2)^*$ with $(KC_p)^* \otimes (KC_p)^*$, $e_{a,b} \mapsto e_a \otimes e_b$.

Lemma 3.1. *Let* $\xi_{1,0} = \xi_1 \otimes 1$ *and* $\xi_{0,1} = 1 \otimes \xi_1 \in (KC_p^2)^*$. *Then*

$$\langle \xi_{1,0}, (\sigma_1 - 1)^j (\sigma_2 - 1)^k \rangle = \delta_{1,j}\delta_{0,k},$$

$$\langle \xi_{0,1}, (\sigma_1 - 1)^j (\sigma_2 - 1)^k \rangle = \delta_{0,j}\delta_{1,k},$$

and $(RC_p^2)^*$ *is an R-Hopf algebra with* $(RC_p^2)^* = R[\xi_{1,0}, \xi_{0,1}]$ *where* $\xi_{1,0}$ *and* $\xi_{0,1}$ *satisfy* $x^p = x$. *On these generators, the counit satisfies* $\varepsilon(x) = 0$, *comultiplication is* $\Delta(x) = x \otimes 1 + 1 \otimes x$, *and the antipode satisfies* $S(x) = -x$.

Define $\wp(x) = x^p - x$.

Proposition 3.2. *Given integers $i_1, i_2 \geq 0$ and $\mu \in K$, let* $\beta_1 = \pi^{i_1}(\xi_{1,0} - \mu\xi_{0,1})$ *and* $\beta_2 = \pi^{i_2}\xi_{0,1}$.

(i) If $v_K(\wp(\mu)) \geq i_2 - pi_1$, then

$$R[\beta_1, \beta_2] = R[\pi^{i_1}(\xi_{1,0} - \mu\xi_{0,1}), \pi^{i_2}\xi_{0,1}]$$

is an R-Hopf order in $(RC_p^2)^*$. The algebra structure of $R[\beta_1, \beta_2]$ is determined by the equations

$$\beta_1^p = \pi^{(p-1)i_1}\beta_1 - \pi^{pi_1 - i_2}\wp(\mu)\beta_2,$$

and

$$\beta_2^p = \pi^{(p-1)i_2}\beta_2.$$

The coalgebra structure of $R[\beta_1, \beta_2]$ is determined on the generators, β_r, $r = 1, 2$, by counit $\varepsilon(\beta_r) = 0$, comultiplication $\Delta(\beta_r) = \beta_r \otimes 1 + 1 \otimes \beta_r$, and antipode $S(\beta_r) = -\beta_r$. In particular, the generators β_1, β_2 are primitive.

(ii) Let $\beta_1' = \pi^{i_1}(\xi_{1,0} - \mu'\xi_{0,1})$ *for some* $\mu' \in K$ *satisfying* $v_K(\wp(\mu')) \geq i_2 - pi_1$*. Then* $R[\beta_1', \beta_2]$ *is a Hopf algebra, and* $R[\beta_1', \beta_2] = R[\beta_1, \beta_2]$ *if and only if* $v_K(\mu' - \mu) \geq i_2 - i_1$*.*

On the dual side, we have

Proposition 3.3. *Let* $i_1, i_2 \geq 0$, $\mu \in K$, $\sigma_1^{[\mu]} = \sum_{i=0}^{p-1} \binom{\mu}{i}(\sigma_1 - 1)^i$, *and let*

$$\mathcal{H}_{i_1,i_2,\mu} = R\left[\frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{\sigma_2\sigma_1^{[\mu]} - 1}{\pi^{i_2}}\right].$$

If $\nu_K(\wp(\mu)) \geq i_2 - pi_1$, *then* $\mathcal{H}_{i_1,i_2,\mu}$ *is a Hopf order in* KC_p^2.

Theorem 3.4. *Let* $\mathcal{H}_{i_1,i_2,\mu}$ *be as in Proposition 3.3, then*

$$\mathcal{H}^*_{i_1,i_2,\mu} = R[\beta_1, \beta_2] = R[\pi^{i_1}(\xi_{1,0} - \mu\xi_{0,1}), \pi^{i_2}\xi_{0,1}].$$

We now show that every Hopf order in $(KC_p^2)^*$ is of the form

$$R[\beta_1, \beta_2] = R[\pi^{i_1}(\xi_{1,0} - \mu\xi_{0,1}), \pi^{i_2}\xi_{0,1}].$$

Recall $C_p^2 = \langle \sigma_1, \sigma_2 \rangle$, and let $\mathcal{H}$ be an R-Hopf order in KC_p^2. Let $C_p^2 \to C_p^2/\langle \sigma_1 \rangle$ denote the canonical surjection with $C_p^2/\langle \sigma_1 \rangle \cong \langle \bar{\sigma}_2 \rangle$ where $\bar{\sigma}_2 = \sigma_2\langle \sigma_1 \rangle$. There exists a short exact sequence

$$R \to \mathcal{H}_{i_1} \to \mathcal{H} \to \mathcal{H}_{i_2} \to R, \tag{1}$$

where $\mathcal{H}_{i_1} = R[(\sigma_1 - 1)/\pi^{i_1}]$ and $\mathcal{H}_{i_2} = R[(\bar{\sigma}_2 - 1)/\pi^{i_2}]$, for some $i_1, i_2 \geq 0$.
We dualize (1) to obtain the short exact sequence

$$R \to \mathcal{H}_{i_2}^* \to \mathcal{H}^* \to \mathcal{H}_{i_1}^* \to R. \tag{2}$$

We next translate into the language of group schemes. Let

$$\mathbb{D}^*_{i_1} = \mathrm{Spec}\ \mathcal{H}^*_{i_1}, \quad \mathbb{D}^* = \mathrm{Spec}\ \mathcal{H}^*, \text{ and } \mathbb{D}^*_{i_2} = \mathrm{Spec}\ \mathcal{H}^*_{i_2}.$$

Classifying all Hopf orders $\mathcal{H}$ in (1), or $\mathcal{H}^*$ in (2), is the same as classifying all finite group schemes $\mathbb{D}^*$ that fit into the short exact sequence of group schemes

$$0 \to \mathbb{D}^*_{i_1} \to \mathbb{D}^* \to \mathbb{D}^*_{i_2} \to 0, \tag{3}$$

and which are represented by an R-Hopf order in $(KC_p^2)^*$. In other words, we compute the subgroup $\mathrm{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{D}^*_{i_1})$ of generically trivial extensions within the full extension group $\mathrm{Ext}^1(\mathbb{D}^*_{i_2}, \mathbb{D}^*_{i_1})$.

To this end, observe that the polynomial ring $R[x]$ with counit $\varepsilon(x) = 0$, comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$ and antipode $S(x) = -x$ represents the additive group scheme $\mathbb{G}_a$.

For $i_1 \geq 0$, the R-algebra map $\psi : R[x] \to R[x]$ determined by $\psi(x) = x^p - \pi^{(p-1)i_1} x$ is a homomorphism of Hopf algebras, and so, there exists a homomorphism of R-group schemes

$$\Psi : \mathbb{G}_a \to \mathbb{G}_a,$$

defined by $\Psi(g)(x) = g(\psi(x))$ for $g \in \mathbb{G}_a$. The kernel of Ψ is represented by the R-Hopf order $R[x]/(\psi(x)) \cong \mathcal{H}_{i_1}^*$ in $(KC_p)^*$, thus there is a short exeact sequence of group schemes

$$0 \to \mathbb{D}_{i_1}^* \xrightarrow{\iota} \mathbb{G}_a \xrightarrow{\Psi} \mathbb{G}_a \to 0. \qquad (4)$$

From (4), we obtain the long exact sequence:

$$\mathrm{Hom}(\mathbb{D}_{i_2}^*, \mathbb{G}_a) \xrightarrow{\Psi} \mathrm{Hom}(\mathbb{D}_{i_2}^*, \mathbb{G}_a) \xrightarrow{\omega} \mathrm{Ext}^1(\mathbb{D}_{i_2}^*, \mathbb{D}_{i_1}^*) \xrightarrow{\iota} \mathrm{Ext}^1(\mathbb{D}_{i_2}^*, \mathbb{G}_a),$$

with connecting homomorphism ω, which induces the map ρ in the exact sequence

$$0 \to \mathrm{coker}(\Psi : \mathrm{Hom}(\mathbb{D}_{i_2}^*, \mathbb{G}_a)^{\circlearrowleft}) \xrightarrow{\rho} \mathrm{Ext}^1(\mathbb{D}_{i_2}^*, \mathbb{D}_{i_1}^*) \xrightarrow{\iota} \mathrm{Ext}^1(\mathbb{D}_{i_2}^*, \mathbb{G}_a).$$

Tensoring with K and considering kernels, we obtain the exact sequence

$$0 \to \mathrm{coker}(\Psi : \mathrm{Hom}(\mathbb{D}_{i_2}^*, \mathbb{G}_a)^{\circlearrowleft})_{gt} \xrightarrow{\rho} \mathrm{Ext}^1_{gt}(\mathbb{D}_{i_2}^*, \mathbb{D}_{i_1}^*) \xrightarrow{\iota} \mathrm{Ext}^1_{gt}(\mathbb{D}_{i_2}^*, \mathbb{G}_a). \tag{5}$$

Proposition 3.5. *There is an isomorphism*

$$\rho : \mathrm{coker}(\Psi : \mathrm{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)^{\circlearrowleft})_{gt} \to \mathrm{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{D}^*_{i_1}).$$

Proof. Our plan is to show that $\mathrm{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{G}_a) = 0$ in (5). To this end, we use a first quadrant spectral sequence to show that $\mathrm{Ext}^1(\mathbb{D}^*_{i_2}, \mathbb{G}_a) \cong \mathrm{H}^2_0(\mathbb{D}^*_{i_2}, \mathbb{G}_a)$. With this characterization, we then form the complex of morphisms

$$\mathrm{Mor}_0((\mathbb{D}^*_{i_2})^{r-1}, \mathbb{X}) \xrightarrow{\partial_{r-1}} \mathrm{Mor}_0((\mathbb{D}^*_{i_2})^{r}, \mathbb{X}) \xrightarrow{\partial_{r}} \mathrm{Mor}_0((\mathbb{D}^*_{i_2})^{r+1}, \mathbb{X}) \xrightarrow{\partial_{r+1}},$$

and compute directly that

$$\mathrm{H}^2_0(\mathbb{D}^*_{i_2}, \mathbb{G}_a) \to \mathrm{H}^2_0(K \otimes_R \mathbb{D}^*_{i_2}, K \otimes_R \mathbb{G}_a)$$

is an injection, thus $\mathrm{H}^2_0(\mathbb{D}^*_{i_2}, \mathbb{G}_a)_{gt} \cong \mathrm{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{G}_a) = 0$ is trivial. $\square$

In order to compute the elements of $\mathrm{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{D}^*_{i_1})$, explicitly, we need to characterize $\mathrm{coker}(\Psi_1 : \mathrm{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)^{\circlearrowleft})_{gt}$.

Proposition 3.6. *The* $\mathrm{coker}(\Psi_1 : \mathrm{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)^{\circlearrowleft})_{gt}$ *is isomorphic to the additive subgroup of* $K/(\mathbb{F}_p + P^{i_2 - i_1})$ *represented by those elements* $\mu \in K$ *satisfying* $\wp(\mu) \in P^{i_2 - pi_1}$.

Proof. Each element of $\mathrm{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)$ corresponds to a R-Hopf algebra homomorphism $R[x] \to \mathcal{H}^*_{i_2}$, and since x is primitive, elements of $\mathrm{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)$ correspond to $\mathrm{Prim}(\mathcal{H}^*_{i_2})$, the primitive elements in $\mathcal{H}^*_{i_2}$. We have $\mathcal{P} = \mathrm{Prim}(\mathcal{H}^*_{i_2}) = R\beta_2$ where $\beta_2 = \pi^{i_2}\xi_{0,1}$.

The generically trivial elements in the cokernel $\mathrm{coker}(\Psi_1 : \mathrm{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)^{\circlearrowleft})$ correspond to elements of

$$(\psi(K \otimes_R \mathcal{P}) \cap \mathcal{P})/\psi(\mathcal{P}).$$

Elements of $K \otimes_R \mathcal{P}$ can be expressed as $\mu\pi^{i_1}\xi_{0,1}$ for some $\mu \in K$, and an element of $\psi(K \otimes_R \mathcal{P})$ can be written

$$\wp(\mu)\pi^{pi_1}\xi_{0,1} = \psi(\mu\pi^{i_1}\xi_{0,1}).$$

An element of $\psi(K \otimes_R \mathcal{P})$ lies in $\mathcal{P}$ precisely when $\wp(\mu) \in P^{i_2 - pi_1}$. It is zero in the quotient $(\psi(K \otimes_R \mathcal{P}) \cap \mathcal{P})/\psi(\mathcal{P})$ precisely when $\mu \in \mathbb{F}_p + P^{i_2 - i_1}$. $\square$

Theorem 3.7. *Each class* $[E]$ *in* $\mathrm{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{D}^*_{i_1})$ *corresponds to a short exact sequence*

$$E_\mu :\ 0 \to \mathbb{D}^*_{i_1} \longrightarrow \mathrm{Spec}\ R[\pi^{i_1}(\xi_{1,0} - \mu\xi_{0,1}), \pi^{i_2}\xi_{0,1}] \longrightarrow \mathbb{D}^*_{i_2} \to 0$$

where $\mu \in K$ *represents a coset in* $K/(\mathbb{F}_p + P^{i_2 - i_1})$ *that satisfies* $\nu_K(\wp(\mu)) \geq i_2 - pi_1$.

Proof. Let $[E] \in \mathrm{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{D}^*_{i_1})$,

$$E :\ 0 \to \mathbb{D}^*_{i_1} \longrightarrow \mathbb{D}^* \longrightarrow \mathbb{D}^*_{i_2} \to 0.$$

By Proposition 3.5, $\rho^{-1}([E]) = [h]$ is a class in the cokernel represented by a homomorphism $h : \mathbb{D}^*_{i_2} \to \mathbb{G}_a$ and is determined by a Hopf algebra map $x \mapsto \wp(\mu)\pi^{pi_1}\xi_{0,1} = \wp(\mu)\pi^{pi_1}\xi_{0,1}$ for some $\mu \in K$ with $\nu_K(\wp(\mu)) \geq i_2 - pi_1$.

We compute the representing Hopf algebra $\mathcal{H}_h^*$ of $\mathbb{D}_h^* = \mathbb{D}^*$.

Translating to Hopf algebras, we have the push-out diagram

$$\begin{array}{ccc} \mathcal{H}_h^* & \leftarrow & R[x] \\ \uparrow & & \psi \uparrow \\ \mathcal{H}_{i_2}^* & \stackrel{\alpha}{\leftarrow} & R[x], \end{array}$$

with $\alpha(x) = \wp(\mu)\pi^{pi_1}\xi_{0,1} = \psi(\mu\pi^{i_1}\xi_{0,1})$. Thus,

$$\begin{aligned} \mathcal{H}_h^* &= (R[\pi^{i_2}\xi_{0,1}] \otimes_R R[x])/(\wp(\mu)\pi^{pi_1}\xi_{0,1} \otimes 1 + 1 \otimes \psi(x)) \\ &\cong R[\pi^{i_2}\xi_{0,1}][x]/(\psi(x) + \wp(\mu)\pi^{pi_1}\xi_{0,1}) \\ &= R[\pi^{i_2}\xi_{0,1}][x]/(\psi(x) + \psi(\mu\pi^{i_1}\xi_{0,1})) \\ &= R[\pi^{i_2}\xi_{0,1}][x]/(\psi(x + \mu\pi^{i_1}\xi_{0,1})). \end{aligned}$$

With $x \mapsto \pi^{i_1}\xi_{1,0}$, under $R[x] \to R[x]/\psi(x) \cong R[\pi^{i_1}\xi_{1,0}]$, one obtains

$$\mathcal{H}_h^* \cong R[\pi^{i_1}(\xi_{1,0} - \mu\xi_{0,1}), \pi^{i_2}\xi_{0,1}].$$

And as we have seen,

$$\begin{aligned}\mathcal{H}_h &\cong R[\pi^{i_1}(\xi_{1,0}-\mu\xi_{0,1}), \pi^{i_2}\xi_{0,1}]^* \\ &\cong \mathcal{H}_{i_1,i_2,\mu} \\ &= R\left[\frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{\sigma_2\sigma_1^{[\mu]} - 1}{\pi^{i_2}}\right].\end{aligned}$$

Thus every R-Hopf order in KC_p^2 is of the form $\mathcal{H}_{i_1,i_2,\mu}$.

4. Hopf orders in $(KC_p^3)^*$

How much of the method of the $n = 2$ case carries over to $n \geq 3$?

Let $C_p^3 = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$, $\bar{\sigma}_2 = \sigma_2 \langle \sigma_1 \rangle$, $\bar{\sigma}_3 = \sigma_3 \langle \sigma_1 \rangle$, and let

$$R \to R\left[\frac{\sigma_1 - 1}{\pi^{i_1}}\right] \to \mathcal{H} \to R\left[\frac{\bar{\sigma}_2 - 1}{\pi^{i_2}}, \frac{\bar{\sigma}_3 \bar{\sigma}_2^{[\mu]} - 1}{\pi^{i_3}}\right] \to R$$

be a short exact sequence of R-Hopf orders, $\mathcal{H} \subseteq KC_p^3$, dualizing as

$$R \to R[\pi^{i_2}(\xi_{0,1,0} - \mu\xi_{0,0,1}), \pi^{i_3}\xi_{0,0,1}] \to \mathcal{H}^* \to R[\pi^{i_1}\xi_{1,0,0}] \to R,$$

where $\xi_{i,j,k} = \xi_i \otimes \xi_j \otimes \xi_k$.

Applying Spec gives

$$0 \to \mathbb{D}^*_{i_1} \to \mathbb{D}^* \to \mathbb{D}^*_{i_2,i_3,\mu} \to 0, \tag{6}$$

where

$$\mathbb{D}^*_{i_2,i_3,\mu} = \mathrm{Spec}\ R[\pi^{i_2}(\xi_{0,1,0} - \mu\xi_{0,0,1}), \pi^{i_3}\xi_{0,0,1}].$$

Note: $\mathbb{D}^*_{i_2,i_3,\mu}$ plays the role of $\mathbb{D}^*_{i_2}$ in the $n = 2$ case.

We want to classify short exact sequences of the form (6). Most of the results in the $n = 2$ case extend easily, in fact:

Proposition 4.1. *There is an isomorphism*

$$\rho : \mathrm{coker}(\Psi : \mathrm{Hom}(\mathbb{D}^*_{i_2,i_3,\mu}, \mathbb{G}_a)^{\circlearrowleft})_{gt} \to \mathrm{Ext}^1_{gt}(\mathbb{D}^*_{i_2,i_3,\mu}, \mathbb{D}^*_{i_1}).$$

So it is a matter of computing $\operatorname{coker}(\Psi : \operatorname{Hom}(\mathbb{D}^*_{i_2,i_3,\mu}, \mathbb{G}_a)^{\circlearrowleft})_{gt}$.

To this end, we see that elements of $\operatorname{Hom}(\mathbb{D}^*_{i_2,i_3,\mu}, \mathbb{G}_a)$ correspond to Hopf maps $R[x] \to R[\pi^{i_2}(\xi_{0,1,0} - \mu\xi_{0,0,1}), \pi^{i_3}\xi_{0,0,1}]$ given as $x \mapsto a$, where $a \in \mathcal{P} = \operatorname{Prim}(R[\pi^{i_2}(\xi_{0,1,0} - \mu\xi_{0,0,1}), \pi^{i_3}\xi_{0,0,1}])$.

Ultimately, we need to compute

$$(\psi(K \otimes \mathcal{P}) \cap \mathcal{P})/\psi(\mathcal{P}).$$

Now, $K \otimes \mathcal{P} = K\xi_{0,1,0} + K\xi_{0,0,1}$, and elements of $K \otimes \mathcal{P}$ can be written

$$\omega\pi^{i_1}\xi_{0,1,0} + \theta\pi^{i_1}\xi_{0,0,1}$$

for $\omega, \theta \in K$.

Thus an element in $\psi(K \otimes \mathcal{P})$ is

$$\psi(\omega\pi^{i_1}\xi_{0,1,0} + \theta\pi^{i_1}\xi_{0,0,1}) = \wp(\omega)\pi^{pi_1}\xi_{0,1,0} + \wp(\theta)\pi^{pi_1}\xi_{0,0,1}.$$

This element is in $\mathcal{P}$ under certain conditions on $\wp(\omega)$ and $\wp(\theta)$; it is in $\psi(\mathcal{P})$ under certain conditions on ω and θ.

We determine these conditions.

Note that $\wp(\omega)\pi^{pi_1}\xi_{0,1,0} + \wp(\theta)\pi^{pi_1}\xi_{0,0,1} \in \mathcal{P}$ if and only if

$$\langle \wp(\omega)\pi^{pi_1}\xi_{0,1,0} + \wp(\theta)\pi^{pi_1}\xi_{0,0,1}, \mathcal{H}_{i_2,i_3,\mu} \rangle \subseteq R.$$

Since

$$
\begin{aligned}
\bar{\sigma}_3\bar{\sigma}_2^{[\mu]} - 1 &= (\bar{\sigma}_3 - 1 + 1)\bar{\sigma}_2^{[\mu]} - 1 \\
&= (\bar{\sigma}_3 - 1)\sum_{i=0}^{p-1}\binom{\mu}{i}(\bar{\sigma}_2 - 1)^i + \sum_{i=1}^{p-1}\binom{\mu}{i}(\bar{\sigma}_2 - 1)^i \\
&= (\bar{\sigma}_3 - 1)\left(1 + \sum_{i=1}^{p-1}\binom{\mu}{i}(\bar{\sigma}_2 - 1)^i\right) \\
&\qquad + \mu(\bar{\sigma}_2 - 1) + \sum_{i=2}^{p-1}\binom{\mu}{i}(\bar{\sigma}_2 - 1)^i \\
&= (\bar{\sigma}_3 - 1) + \mu(\bar{\sigma}_2 - 1) + \sum_{i=2}^{p-1}\binom{\mu}{i}(\bar{\sigma}_2 - 1)^i \\
&\qquad + \sum_{i=1}^{p-1}\binom{\mu}{i}(\bar{\sigma}_3 - 1)(\bar{\sigma}_2 - 1)^i,
\end{aligned}
$$

It suffices to show that

$$\langle \wp(\omega)\pi^{pi_1}\xi_{0,1,0} + \wp(\theta)\pi^{pi_1}\xi_{0,0,1}, \bar{\sigma}_2 - 1\rangle \in \pi^{i_2}R,$$

and

$$\langle \wp(\omega)\pi^{pi_1}\xi_{0,1,0} + \wp(\theta)\pi^{pi_1}\xi_{0,0,1}, (\bar{\sigma}_3 - 1) + \mu(\bar{\sigma}_2 - 1)\rangle \in \pi^{i_3}R,$$

The first condition is

$$\nu_K(\wp(\omega)) \geq i_2 - pi_1,$$

and the second condition is

$$\nu(\wp(\theta) + \mu\wp(\omega)) \geq i_3 - pi_1.$$

Note: if $\nu_K(\mu) \leq 0$, then $\nu_K(\mu) \geq \dfrac{i_3}{p} - i_2$. Thus,

$$\nu_K(\mu\wp(\omega)) \geq \frac{i_3}{p} - i_2 + i_2 - pi_1 = \frac{i_3}{p} - pi_1,$$

and so,

$$\nu(\wp(\theta)) \geq \frac{i_3}{p} - pi_1.$$

Here is the classification result.

Theorem 4.2. *Each class* $[E]$ *in* $\mathrm{Ext}^1_{gt}(\mathbb{D}^*_{i_2,i_3,\mu}, \mathbb{D}^*_{i_1})$ *corresponds to a short exact sequence*

$$E_{\omega,\theta}: \; 0 \to \mathbb{D}^*_{i_1}$$
$$\longrightarrow \mathrm{Spec}\, R[\pi^{i_1}(\xi_{1,0,0} - \omega\xi_{0,1,0} - \theta\xi_{0,0,1}), \pi^{i_2}(\xi_{0,1,0} - \mu\xi_{0,0,1}), \pi^{i_3}\xi_{0,0,1}]$$
$$\longrightarrow \mathbb{D}^*_{i_2,i_3,\mu} \to 0$$

where $\mu, \omega, \theta \in K$ *satisfy*

$$\nu_K(\wp(\mu)) \geq i_3 - pi_2, \quad \nu(\wp(\omega)) \geq i_2 - pi_1, \quad \nu_K(\wp(\theta)) \geq \frac{i_3}{p} - pi_1.$$

Finally, we have a conjecture.

Conjecture 4.3. *The Hopf order*

$$R[\pi^{i_1}(\xi_{1,0,0} - \omega\xi_{0,1,0} - \theta\xi_{0,0,1}), \pi^{i_2}(\xi_{0,1,0} - \mu\xi_{0,0,1}), \pi^{i_3}\xi_{0,0,1}]$$

in $(KC_p^3)^*$ *is the linear dual of the Hopf order*

$$R\left[\frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{\sigma_2\sigma_1^{[\omega]} - 1}{\pi^{i_2}}, \frac{\sigma_3\sigma_1^{[\theta]}(\sigma_2\sigma_1^{[\omega]})^{[\mu]} - 1}{\pi^{i_3}}\right]$$

in KC_p^3.

References

1. Elder, G. G., Underwood, R. G., Finite group scheme extensions, and Hopf orders in KC_p^2 over a characteristic p discrete valuation ring, *New York J. Math.*, 23, (2017), 1-29.

2. Koch, A., Primitively generated Hopf orders in characteristic p,

3. Tate J., and Oort, F., Group schemes of prime order, Ann. Sci. École Norm. Sup. (4) 3 (1970), 1-21.

4. Tossici, D., Models of $\mu_{p^2,K}$ over a discrete valuation ring, with an appendix by X. Caruso, *J. Algebra*, 323 (2010), no. 7, 1908-1957.