Stable and semistable Hopf-Galois extensions

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Local fields

Let K be a field which is complete with respect to a discrete valuation $v_K : K \to \mathbb{Z} \cup \{\infty\}$.

Assume that the residue field $\overline{K}$ of K is a perfect field of characteristic p. Also let

$$\begin{aligned}
\mathcal{O}_K &= \{\alpha \in K : v_K(\alpha) \geq 0\} \\
&= \text{ring of integers of } K \\[1em]
\pi_K &= \text{uniformizer for } \mathcal{O}_K \text{ (i.e., } v_K(\pi_K) = 1) \\[1em]
\mathcal{M}_K &= \pi_K \mathcal{O}_K \\
&= \text{unique maximal ideal of } \mathcal{O}_K
\end{aligned}$$

Let L/K be a separable totally ramified extension of degree p^n.

Extend $v_L : L \to \mathbb{Z} \cup \{\infty\}$ to $v_L : L^{sep} \to \mathbb{Q}$.

Hopf-Galois extensions

$$\begin{aligned} H &= \text{Hopf algebra over } K \\ L/K &= \text{finite separable } H\text{-Galois extension} \\ E/K &= \text{Galois closure of } L/K \\ G &= \mathrm{Gal}(E/K) \\ G' &= \mathrm{Gal}(E/L) \\ X &= G/G' \\ XE &= Map(X, E) \end{aligned}$$

Extending the base field from K to E gives an E-algebra isomorphism $E \otimes_K L \cong XE$.

Since L/K is an H-Galois extension there is a regular subgroup $N \leq Perm(X)$ and an isomorphism of E-Hopf algebras $E \otimes_K H \cong EN$.

Identify L with $K \otimes_K L \subset E \otimes_K L$.
Identify H with $K \otimes_K H \subset E \otimes_K H$.

The trace element

Set $\theta = \sum_{\eta \in N} \eta \in EN \cong E \otimes_K H$.

Since G normalizes N we get

$$\theta \in (EN)^G \cong (E \otimes_K H)^G = K \otimes_K H.$$

For $\eta \in N$ we have $\eta\theta = \theta\eta = \theta$. Hence $h\theta = \theta h = \epsilon(\theta)h$ for all $h \in H$. It follows that θ is both a left integral and a right integral for H.

Let $\lambda \in L$. Since N acts simply transitively on the set G/G' of K-embeddings of L into E we get

$$\theta(\lambda) = \sum_{\eta \in N} \eta(1G')(\lambda) = \mathrm{Tr}_{L/K}(\lambda).$$

The map $\phi : L \otimes_K L \to L \otimes_K H$

Write $\Delta(\theta) = \sum \theta_{(1)} \otimes \theta_{(2)}$ and define

$$\phi : L \otimes_K L \longrightarrow L \otimes_K H$$
$$\phi(a \otimes b) = \sum a\theta_{(1)}(b) \otimes \theta_{(2)}.$$

Proposition

ϕ is an isomorphism of K-vector spaces.

Let

$$\Delta_E : EN \longrightarrow EN \otimes_E EN$$
$$\phi_E : XE \otimes_E XE \longrightarrow XE \otimes_E EN$$

be the maps induced by $\mathrm{id}_E \otimes \Delta$ and $\mathrm{id}_E \otimes \phi$. Then for $\eta \in N$ we have $\Delta_E(\eta) = \eta \otimes \eta$. Hence for $a, b \in XE$ we get $\phi_E(a \otimes b) = \sum_{\eta \in N} a\eta(b) \otimes \eta$.

A partial order on $(\mathbb{Z} \times \mathbb{Z})/A$

Let A be the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by the element $(p^n, -p^n)$.

For $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ write $[a, b]$ for the coset $(a, b) + A$.

Define a partial order on $(\mathbb{Z} \times \mathbb{Z})/A$ by $[a, b] \leq [c, d]$ if and only if there is $(c', d') \in [c, d]$ such that $a \leq c'$ and $b \leq d'$.

We use the following set of coset representatives for $(\mathbb{Z} \times \mathbb{Z})/A$:

$$\mathcal{F} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq b < p^n\}$$

Let $\mathcal{T}$ be the set of Teichmüller representatives of K, and let $\beta \in L \otimes_K L$. Then there are unique $d_{ij} \in \mathcal{T}$ such that

$$\beta = \sum_{(i,j) \in \mathcal{F}} d_{ij} \pi_L^i \otimes \pi_L^j.$$

Set

$$R(\beta) = \{[i, j] : (i, j) \in \mathcal{F},\ d_{ij} \neq 0\}.$$

Diagrams

Definition

Define the diagram of $\beta \in L \otimes_K L$ to be

$$D(\beta) = \{[a,b] \in (\mathbb{Z} \times \mathbb{Z})/A : [i,j] \le [a,b] \text{ for some } [i,j] \in R(\beta)\}.$$

Proposition ([Bon2], Remark 2.4.3)

$D(\beta)$ does not depend on the choice of uniformizer π_L for L.

For $\beta \in L \otimes_K L$ with $\beta \neq 0$ define

$$d(\beta) = \min\{i + j : [i,j] \in D(\beta)\}.$$

Define the diagonal of β to be

$$N(\beta) = \{[i,j] \in D(\beta) : i + j = d(\beta)\}.$$

H-stable and H-semistable extensions

Let $G(\beta)$ denote the set of minimal elements of $D(\beta)$ with respect to the partial order $\leq$. Then $N(\beta) \subset G(\beta)$.

Definition

Let L/K be a totally ramified H-Galois extension of degree p^n.

1. Say that L/K is an H-semistable extension if there is $\beta \in L \otimes_K L$ such that $\phi(\beta) \in H$, $p \nmid d(\beta)$, and $|N(\beta)| = 2$.
2. Say that L/K is an H-stable extension if L/K is H-semistable and we may choose β so that $G(\beta) = N(\beta)$.

A function

Let $\delta_{L/K}$ denote the different of L/K and set $i_0 = v_L(\delta_{L/K}) - p^n + 1$.

For $\xi \in H \smallsetminus \{0\}$ define $f_\xi : \mathbb{Z} \to \mathbb{Z}$ by

$$f_\xi(a) = \min\{v_L(\xi(y)) : y \in \mathcal{M}_L^a\}.$$

Then $f_\xi(a+1) \geq f_\xi(a)$. Furthermore, for every $a \in \mathbb{Z}$ there is $z \in L$ with $v_L(z) = a$ and $v_L(\xi(z)) = f_\xi(a)$.

Recall that $\theta \in H$ is the trace element. We get

$$f_\theta(a) = p^n \left\lceil \frac{a + i_0}{p^n} \right\rceil.$$

It follows that $f_\theta(-i_0) = 0$ and $f_\theta(-i_0 + 1) = p^n$. Hence if $\rho \in L$ with $v_L(\rho) = -i_0$ then $v_L(\theta(\rho)) = 0$.

A fundamental theorem

Theorem

Let $\xi \in H \smallsetminus \{0\}$, let $\beta \in L \otimes_K L$ satisfy $\xi = \phi(\beta)$, and let $a, b \in \mathbb{Z}$. Then the following are equivalent:

1. $[a, b] \in D(\beta)$.
2. $f_\xi(-b - i_0) \leq a$.

Corollary

Let $\beta \in L \otimes_K L$ be such that $\xi := \phi(\beta) \in H$. Then for all $y \in L^\times$ we have $v_L(\xi(y)) \geq v_L(y) + d(\beta) + i_0$, with equality if and only if $v_L(y) \equiv -b - i_0 \pmod{p^n}$ for some $[a, b] \in N(\beta)$.

Proof of the fundamental theorem

Write

$$\beta = \sum_{(i,j)\in\mathcal{F}} d_{ij}\pi_L^i \otimes \pi_L^j.$$

Then for $\lambda \in L$ we have

$$\phi(\beta)(\lambda) = \sum_{(i,j)\in\mathcal{F}} d_{ij}\pi_L^i\theta(\pi_L^j\lambda).$$

Suppose $[a, b] \in D(\beta)$. Then there is $[a', b'] \in G(\beta)$ such that $[a', b'] \leq [a, b]$. We may assume that $(a', b') \in \mathcal{F}$, $a' \leq a$, and $b' \leq b$. Then $d_{a'b'} \neq 0$ and $d_{ij} = 0$ for all $(i, j) \in \mathcal{F}$ such that $(i, j) \neq (a', b')$ and $[i, j] \leq [a', b']$.

Proof of the fundamental theorem ...

Let $y \in L$ satisfy $v_L(y) = -b' - i_0$. Then $v_L(\pi_L^{a'}\theta(\pi_L^{b'}y)) = a'$.

Suppose $(i,j) \in \mathcal{F}$, $(i,j) \neq (a', b')$, and $d_{ij} \neq 0$. Then

$$v_L(\pi_L^i\theta(\pi_L^j y)) \geq i + p^n \left\lceil \frac{j - b'}{p^n} \right\rceil .$$

We have either $i > a'$ or $j > b'$. If $i > a'$ then $v_L(\pi_L^i\theta(\pi_L^j y)) \geq i > a'$. If $j > b'$ then since $i > a' - p^n$ we get $v_L(\pi_L^i\theta(\pi_L^j y)) \geq i + p^n > a'$.

It follows that $v_L(\xi(y)) = a'$. Since $y \in \mathcal{M}_L^{-b-i_0}$ we get $f_\xi(-b - i_0) \leq a' \leq a$.

Suppose $[a, b] \notin D(\beta)$. Let $(i,j) \in \mathcal{F}$ satisfy $d_{ij} \neq 0$. Then $[i,j] \not\leq [a,b]$, and hence $[a+1, b-p^n+1] \leq [i,j]$. By choosing an appropriate representative for $[a,b]$ we may assume that $a+1 \leq i$ and $b - p^n + 1 \leq j$. Let $y \in L$ with $v_L(y) \geq -b - i_0$. Then $v_L(\theta(\pi^j y)) \geq 0$, so $v_L(\pi_L^i\theta(\pi^j y)) \geq i > a$. Hence $f_\xi(-b - i_0) > a$.

Numerical properties of H-semistable extensions

Theorem

Let L/K be an H-semistable extension and let $\beta \in L \otimes_K L$ be the corresponding tensor. Then there is $h \in \mathbb{Z}$ with $h \equiv i_0 \pmod{p^n}$ such that $N(\beta) = \{[0, h], [h, 0]\}$.

Hence by replacing β with a K-multiple we may assume that $N(\beta) = \{[0, i_0], [i_0, 0]\}$.

Since we are not assuming that L/K is Galois, the lower ramification breaks ℓ_i of L/K need not be integers. We do, however, have $\ell_i \in \mathbb{Z}_{(p)}$.

Theorem

Let L/K be an H-semistable extension of degree p^n. Let $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_n$ be the lower ramification breaks of L/K, counted with multiplicity. Then $\ell_i \equiv -i_0 \pmod{p^n \mathbb{Z}_{(p)}}$ for $1 \leq i \leq n$.

Some steps in the right direction

Lemma

There exists ν in the center of H and $h \in \mathbb{N}$ such that $h \equiv -i_0 \pmod{p}$, and for all $\lambda \in L^\times$ we have

$$v_L(\nu(\lambda)) = v_L(\lambda) + h \text{ if } p \nmid v_L(\lambda)$$
$$v_L(\nu(\lambda)) > v_L(\lambda) + h \text{ if } p \mid v_L(\lambda).$$

Note that if $H = K[G]$ then we can take $\nu = \sigma - 1$ for any $\sigma \in Z(G)$ such that $\sigma \neq 1$.

Proposition

Set $c = d(\beta)$ and write $N(\beta) = \{[a_1, c - a_1], [a_2, c - a_2]\}$. Assume that $p \nmid c - a_1$. Then $a_2 \equiv a_1 - h \pmod{p^n}$.

$N(\beta) = \{[a_1, c - a_1], [a_2, c - a_2]\} \Rightarrow a_2 \equiv a_1 - h \pmod{p^n}$

Set $\xi = \phi(\beta)$. It follows from the corollary that for all $\lambda \in L^\times$ we have $v_L(\xi(\lambda)) \geq v_L(\lambda) + c + i_0$, with equality if and only if either $v_L(\lambda) \equiv -c + a_1 - i_0 \pmod{p^n}$ or $v_L(\lambda) \equiv -c + a_2 - i_0 \pmod{p^n}$.

Let $y \in L$ satisfy $v_L(y) = -i_0 - h - c + a_1$. Then $v_L(y) \equiv -c + a_1 \pmod{p}$, so $p \nmid v_L(y)$. Therefore $v_L(\nu(y)) = -i_0 - c + a_1$, so we get $v_L(\xi(\nu(y))) = a_1$.

Since $\xi \circ \nu = \nu \circ \xi$ we get $v_L(\nu(\xi(y))) = a_1$, and hence $v_L(\xi(y)) \leq a_1 - h$. We also have

$$v_L(\xi(y)) \geq v_L(y) + c + i_0 = a_1 - h.$$

Hence $v_L(\xi(y)) = a_1 - h$ for all $y \in L$ such that $v_L(y) = -i_0 - h - c + a_1$.

It follows by the Fundamental Theorem that $[a_1 - h, c - a_1 + h] \in N(\beta)$. Therefore

$$[a_1 - h, c - a_1 + h] = [a_2, c - a_2].$$

We conclude that $a_1 - h \equiv a_2 \pmod{p^n}$.

ϕ and the switch map

Lemma

Let $\beta \in L \otimes_K L$ and let $s : L \otimes_K L \to L \otimes_K L$ be the switch map. If $\phi(\beta) \in H$ then $\phi(s(\beta)) \in H$.

Proof: Let $\alpha \in (E \otimes_K L) \otimes_E (E \otimes_K L)$. It suffices to show that if $\phi_E(\alpha)$ lies in

$$(E \otimes_K K) \otimes_E (E \otimes_K H) \cong E[N] \subset (E \otimes_K L)[N]$$

then so does $\phi_E(s_E(\alpha))$. Write $\alpha = \sum_{i=1}^r a_i \otimes b_i$ with $a_i, b_i \in E \otimes_K L$. Then

$$\phi_E(\alpha) = \sum_{i=1}^r \sum_{\eta \in N} a_i \eta(b_i)\eta = \sum_{\eta \in N} \psi_\eta(\alpha)\eta,$$

with $\psi_\eta(\alpha) = \sum_{i=1}^r a_i \eta(b_i) \in E$. Hence

$$\psi_\eta(s_E(\alpha)) = \sum_{i=1}^r b_i \eta(a_i) = \eta\left(\sum_{i=1}^r a_i \eta^{-1}(b_i)\right) = \eta(\psi_{\eta^{-1}}(\alpha)) = \psi_{\eta^{-1}}(\alpha).$$

It follows that $\psi_\eta(s_E(\alpha)) \in E$. Therefore $\phi_E(s_E(\alpha)) \in E[N]$.

Some isomorphisms of $\mathfrak{A}_0$-modules

For $\xi \in H \setminus \{0\}$ define

$$\hat{v}_L(\xi) = \min\{v_L(\xi(\lambda)) - v_L(\lambda) : \lambda \in L^\times\}.$$

For $h \in \mathbb{Z}$ define

$$\mathfrak{A}_h = \{\xi \in H : \hat{v}_L(\xi) \geq h\}.$$

Let $f \in \mathfrak{A}_h$ and $g \in \mathfrak{A}_k$. Then $f \circ g \in \mathfrak{A}_{h+k}$. It follows that $\mathfrak{A}_0$ is a $\mathcal{O}_K$-algebra, and that $\mathfrak{A}_h$ is a left and right $\mathfrak{A}_0$-module for all $h \in \mathbb{Z}$.

Theorem

Let L/K be an H-semistable extension and let $h \in \mathbb{Z}$. Then for every $\rho \in L$ such that $v_L(\rho) = -i_0$ we have $\mathfrak{A}_{h+i_0} \cdot \rho = \mathcal{M}_L^h$. Hence there is an isomorphism of $\mathfrak{A}_0$-modules $\mathfrak{A}_{h+i_0} \cong \mathcal{M}_L^h$.

Corollary

Let L/K be an H-semistable extension. Then $\mathcal{M}_L^{-i_0}$ is free over its associated H-order $\mathfrak{A}(\mathcal{M}_L^{-i_0})$.

A basis for H

Proposition

Let $\alpha, \beta \in L \otimes_K L$ be such that $\phi(\alpha) \in H$ and $\phi(\beta) \in H$. Then $\phi(\alpha\beta) \in H$.

Corollary

Let $\beta \in L \otimes_K L$ satisfy $\phi(\beta) \in H$. Then for all $s \in \mathbb{S}_{p^n}$ we have $\phi(\beta^s) \in H$.

Let L/K be an H-semistable extension. Then there is $\beta \in L \otimes_K L$ such that $\phi(\beta) \in H$ and $N(\beta) = \{[0, i_0], [i_0, 0]\}$. It follows that for $s \in \mathbb{S}_{p^n}$ we have $d(\beta^s) = si_0$ and $N(\beta^s) = \{[ji_0, (s-j)i_0] : j \preceq s\}$.

Set $\xi^{*s} = \phi(\beta^s)$. Then $\xi^{*s} \in H$. For $y \in L^\times$ we get $v_L(\xi^{*s}(y)) \geq v_L(y) + (s+1)i_0$, with equality if and only if $v_L(y) \equiv -(j+1)i_0 \pmod{p^n}$ for some j such that $j \preceq s$.

The set $\{\xi^{*s} : s \in \mathbb{S}_{p^n}\}$ is a K-basis for H.

Hopf-Galois module structures

For $g, h \in \mathbb{Z}$ and $s \in \mathbb{S}_{p^n}$ define

$$c(g,h) = \left\lfloor \frac{gi_0 - h}{p^n} \right\rfloor$$

$$w(s,h) = \min\{c(s-j,h) - c(-j-1,h) : j \preceq s\}.$$

Theorem (cf. [BCE, Theorem 3.1])

Let L/K be an H-stable extension of degree p^n, let $h \in \mathbb{Z}$, and let $\beta \in L \otimes_K L$ satisfy $\phi(\beta) \in K \otimes_K H$ and $G(\beta) = \{[0, i_0], [i_0, 0]\}$. Then

1. *An $\mathcal{O}_K$-basis for the associated order $\mathfrak{A}(\mathcal{M}_L^h)$ of $\mathcal{M}_L^h$ is given by*

$$S = \{\pi_K^{-w(s,h)} \phi(\beta^s) : 0 \leq s < p^n\}.$$

2. *If $w(s,h) = c(s,h) - c(-1,h)$ for all $s \in \mathbb{S}_{p^n}$ then $\mathcal{M}_L^h$ is free over $\mathfrak{A}(\mathcal{M}_L^h)$.*
3. *If $\mathcal{M}_L^h$ is free over $\mathfrak{A}(\mathcal{M}_L^h)$ and $\mathfrak{A}(\mathcal{M}_L^h)$ is a local ring then $w(s,h) = c(s,h) - c(-1,h)$ for all $s \in \mathbb{S}_{p^n}$.*

Scaffolds and H-semistable extensions

Theorem

Let L/K be a semistable extension of degree p^n. Then L/K has an H-scaffold with precision 1.

Let $h \in \mathbb{S}_{p^n}$ satisfy $h \equiv i_0 \pmod{p^n}$ and set

$$m_{L/K} = \max\{h-1, p^n-h-1\}.$$

Theorem

Let L/K be a totally ramified H-Galois extension of degree p^n.

1. *If L/K has an H-scaffold of precision $\mathfrak{c} \geq 1$ then L/K is H-semistable.*
2. *If L/K has an H-scaffold of precision $\mathfrak{c} \geq m_{L/K}$ then L/K is H-stable.*

An application

Let L/K be an H-Galois extension with lower ramification breaks $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_n$.

Suppose L/K has an H-scaffold $(\{\Psi_i\}, \{y_t\})$. Then L/K is H-semistable, so there is $\beta \in L \otimes_K L$ such that $\phi(\beta) \in H$ and $N(\beta) = \{[0, i_0], [i_0, 0]\}$.

Hence there is an H-scaffold $(\{\Psi'_i\}, \{y'_t\})$ for L/K such that $\Psi'_i = \phi(\beta^{p^n - p^{n-i} - 1})$ for $1 \leq i \leq n$. Let b'_i be the shift associated to Ψ'_i. We get

$$\begin{aligned} p^{n-i} b'_i &= (p^n - p^{n-i}) i_0 \\ b'_i &= (p^i - 1) i_0 \\ b'_i &\equiv -i_0 && \pmod{p^i} \\ b'_i &\equiv \ell_i && \pmod{p^i \mathbb{Z}_{(p)}}. \end{aligned}$$

Extensions of degree p (Hopf-Galois structures)

L/K = separable totally ramified extension of degree p.

E/K = Galois closure of L/K, $G = \mathrm{Gal}(E/K)$, $G' = \mathrm{Gal}(E/L)$.

Let G_1 be the wild ramification subgroup of G. Then $G_1 \trianglelefteq G$, so $N := \lambda(G_1)$ is normalized by $\lambda(G)$. Since $|N| = |G_1| = p$ and $p \nmid |G'|$, N acts simply transitively on G/G' by left multiplication. Hence there is a Hopf-Galois structure on L/K associated to N.

Suppose N' is another regular subgroup of $Perm(G/G')$ which is normalized by $\lambda(G)$. Then $\lambda(G)$ is contained in the holomorph $Hol(N')$ of N'. Since $|N'| = |G/G'| = p$ is prime, the only subgroup of order p of $Hol(N')$ is N'. Since $N \leq \lambda(G) \leq Hol(N')$ and $|N| = p$ we get $N' = N$.

Conclude that L/K has a unique Hopf-Galois structure.

Extensions of degree p (constructing a scaffold)

(This is done in [Degp].)

Assume that $i_0 < v_L(p)$.

Let σ be a generator for $G_1 \cong C_p$ and let γ be a generator for $G' \cong C_d$.

Then $\gamma\sigma\gamma^{-1} = \sigma^r$ for some $r \in \mathbb{Z}$ such that $r + p\mathbb{Z}$ has order d in $(\mathbb{Z}/p\mathbb{Z})^\times$. Hence there is a primitive dth root of unity $\zeta_d \in K$ such that $r \equiv \zeta_d \pmod{\mathcal{M}_K}$.

Let $M = E^{G_1}$. Then $\mathrm{Gal}(E/L) \cong \mathrm{Gal}(M/K)$, so there is $\alpha \in \mathcal{O}_M$ such that $\gamma(\alpha)/\alpha = \zeta_d^{-1}$. Set

$$\Psi_1 = \alpha \cdot \sum_{i=0}^{d-1} \zeta_d^{-i} \eta^{r^i}.$$

Extensions of degree p (constructing a scaffold ...)

We get $\Psi_1(1) = 0$ and $\sigma \cdot \Psi_1 = \gamma \cdot \Psi_1 = \Psi_1$. Hence $\Psi_1 \in (EN)^G = H$.

Let I denote the augmentation ideal of $\mathcal{O}_E N$. We find that

$$\Psi_1 \equiv d\alpha(\eta - 1) \pmod{\alpha I^2}.$$

Let ℓ denote the ramification break of L/K and let $y \in L^\times$. It follows from the congruence above that

$$\begin{aligned} v_L(\Psi_1(y)) &= v_L(\alpha(\eta(y) - y)) \\ &\geq v_L(\alpha) + v_L(y) + \ell, \end{aligned}$$

with equality if and only if $p \nmid v_L(y)$. Set $b = v_L(\alpha) + \ell$. Then $b = v_L(\Psi_1(\pi_L)) - 1 \in \mathbb{Z}$. Since $\alpha \in M$ we get $b \equiv \ell \pmod{p\mathbb{Z}_{(p)}}$.

Extensions of degree p (constructing a scaffold......)

Let $\rho \in L$ satisfy $v_L(\rho) = b$, and for $t \in \mathbb{Z}$ let $c_t \in \mathbb{S}_p$ be such that $bc_t \equiv t - b \pmod{p}$. Set

$$f_t = (t - b - bc_t)/p, \qquad \lambda_t = \pi_K^{f_t} \Psi_1^{c_t}(\rho).$$

Then $v_L(\lambda_t) = t$ and $\lambda_{t_1}\lambda_{t_2}^{-1} \in K$ when $t_1 \equiv t_2 \pmod{p}$. Furthermore, $\Psi_1(\lambda_t) = \lambda_{t+b}$ for all $t \in \mathbb{Z}$ such that $p \nmid t$.

We also have $\Psi_1^p/\alpha^p \in (p\mathcal{O}_E N) \cap I = pI$, so we get

$$\begin{aligned} v_L(\Psi_1(\lambda_{pb})) &= v_L(\Psi_1(\Psi_1^{p-1}(\rho))) \\ &\geq v_L(p) + p v_L(\alpha) + v_L(\rho) + \ell \\ &= pb + b + (v_L(p) - (p-1)\ell). \end{aligned}$$

Setting

$$\mathfrak{c} = v_L(p) - (p-1)\ell = v_L(p) - i_0 > 0$$

we get $v_L(\Psi_1(\lambda_{ps})) \geq ps + b + \mathfrak{c}$ for all $s \in p\mathbb{Z}$. Hence $(\{\Psi_1\}, \{\lambda_t\}_{t\in\mathbb{Z}})$ is an H-scaffold for L/K with precision $\mathfrak{c}$.

Extensions of degree p (semistable and stable)

Set $\xi = \Psi_1^{p-2}$. Then for $t \in \mathbb{Z}$ we have

$$\begin{aligned} \xi(\lambda_t) &= \lambda_{t+(p-2)b} && \text{if } t \equiv b \pmod{p}, \\ \xi(\lambda_t) &= \lambda_{t+(p-2)b} && \text{if } t \equiv 2b \pmod{p}, \\ v_L(\xi(\lambda_t)) &\geq t + (p-2)b + \mathfrak{c} && \text{otherwise.} \end{aligned}$$

Let $\beta \in L \otimes_K L$ be such that $\phi(\beta) = \xi$. Then $d(\beta) = (p-2)b - i_0$, and for all $[x, y] \in G(\beta) \smallsetminus N(\beta)$ we have $x + y \geq d(\beta) + \mathfrak{c}$. Hence L/K is H-semistable with precision $\mathfrak{c}$.

It follows that if $\mathfrak{c} \geq m_{L/K}$ then L/K is H-stable.

Some questions

1. What about those formal group laws?
2. Can these constructions be extended to inseparable extensions?