Isomorphism problems on Hopf algebras and braces

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Outline

Setup

Let L/K be a Galois extension, group G.

Let $N_1, N_2 \leq \mathrm{Perm}(G)$ be regular, G-stable subgroups.

Let $H_1 = (L[N_1])^G,\ H_2 = (L[N_2])^G$.

Let $\mathfrak{B}_1 = \mathfrak{B}(N_1), \mathfrak{B}_2 = \mathfrak{B}(N_2)$ be the corresponding (skew left) braces.

Two possible isomorphism questions:

1. Are $H_1 \cong H_2$ as K-Hopf algebras?
2. Are $\mathfrak{B}_1 \cong \mathfrak{B}_2$ as braces?

Also: what, if any, relation is there between the answers to (1) and (2)?

Hopf algebra isomorphism problem

$N_1, N_2 \leq \mathrm{Perm}(G),\ H_1 = (L[N_1])^G,\ H_2 = (L[N_2])^G.$

Recall G acts on N_i by conjugation by left translation, i.e.,

$$^g\eta = \lambda(g)\eta\lambda(g^{-1}).$$

Theorem (TARP, 2019)

$H_1 \cong H_2$ iff there exists a G-equivariant isomorphism $\theta : N_1 \to N_2$.

So we require $\theta(\,^g\eta) = \,^g\theta(\eta)$ for all $g \in G, \eta \in N$.

Brace isomorphism problem

Recall that the correspondence:

$$\{N \leq \mathrm{Perm}(G) \text{ regular, } G\text{-stable}\} \Rightarrow \{\text{iso. classes } (B,\cdot,\circ),\ (B,\circ) \cong G\}$$
$$N \to [\mathfrak{B}(N)]$$

is surjective but not injective.

Can we determine whether $[\mathfrak{B}(N_1)] = [\mathfrak{B}(N_2)]$?

Or, given N, can we construct all regular, G-stable $M \leq \mathrm{Perm}(G)$ with $\mathfrak{B}(M) \cong \mathfrak{B}(N)$?

We will focus on the latter question (whose answer is known).

First approach

Let $\mathfrak{B} := \mathfrak{B}(N) = (B, \cdot, \circ)$; and let $\varphi \in \mathrm{Aut}(B, \circ) \cong \mathrm{Aut}(G)$.

Define a binary operation $\cdot_\varphi$ on B by

$$x \cdot_\varphi y = \varphi^{-1}(\varphi(x) \cdot \varphi(y)).$$

Then $(B, \cdot_\varphi)$ is a group. Moreover,

$$\begin{aligned} x \circ (y \cdot_\varphi z) &= x \circ \varphi^{-1}(\varphi(y) \cdot \varphi(z)) \\ &= \varphi^{-1}\left(\varphi(x) \circ (\varphi(y) \cdot \varphi(z))\right) \\ &= \varphi^{-1}\Big((\varphi(x) \circ \varphi(y)) \cdot \varphi(x^{-1}) \cdot (\varphi(x) \circ \varphi(z))\Big) \\ &= \varphi^{-1}\Big((\varphi(x \circ y)) \cdot \varphi(x^{-1}) \cdot (\varphi(x \circ z))\Big) \\ &= (x \circ y) \cdot_\varphi x^{-1} \cdot_\varphi (x \circ z), \end{aligned}$$

hence $\mathfrak{B}_\varphi := (B, \cdot_\varphi, \circ)$ is a brace.

$(B, \cdot, \circ)$ vs $(B, \cdot_\varphi, \circ)$

Recall the construction of $\mathfrak{B}(N)$: $(B, \cdot) = N$, and

$$x \circ y = a^{-1}(a(x) *_G a(y))$$

where $a : N \to G$ is given by $a(\eta) = \eta[1_G]$.

Given an arbitrary $(B, \cdot, \circ)$ we can construct an $N \leq \mathrm{Perm}(B, \circ)$. Set $G = (B, \circ)$, and let $N \leq \mathrm{Perm}(G)$ be given by

$$\eta[g] = \eta \cdot g, \; g \in G, \eta \in N.$$

In this setting, $(B, \cdot_\varphi, \circ)$ corresponds to the regular subgroup $N_\varphi \leq \mathrm{Perm}(G, \circ)$ with

$$\eta_\varphi[g] = \eta \cdot_\varphi g = \varphi^{-1}(\varphi(\eta) \cdot \varphi(g)),$$

and $N_\varphi \neq N$ unless φ respects $\cdot$, i.e., is a brace automorphism.

Second approach

Given $N \leq \text{Perm}(G)$, $\varphi \in \text{Aut}(G)$, define

$$N_\varphi = \varphi N \varphi^{-1}.$$

Since $\varphi\eta\varphi^{-1}[g] = g$ iff $\eta[\varphi^{-1}(g)] = \varphi^{-1}(g)$, $N_\varphi \leq \text{Perm}(G)$ is regular.
Also, N_φ is G-stable:

$$\begin{aligned}
\lambda(g)N_\varphi\lambda(g^{-1}) &= \lambda(g)\varphi N\varphi^{-1}\lambda(g^{-1}) \\
&= \varphi\left(\lambda(\varphi^{-1}(g))N\lambda(\varphi^{-1}(g^{-1}))\right)\varphi^{-1} \\
&= \varphi\left(\lambda(h)N\lambda(h^{-1})\right)\varphi^{-1},\ h = \varphi^{-1}(g) \\
&= \varphi N\varphi^{-1} \\
&= N_\varphi.
\end{aligned}$$

Long story short

[Byott] The number of Hopf-Galois structures corresponding to a given brace $(B, \cdot, \circ)$ is

$$|\mathrm{Aut}(B, \circ)/\,\mathrm{Aut}(B, \cdot, \circ)|\,.$$

[Zenouz] Given a regular, G-stable subgroup $N \leq \mathrm{Perm}(G)$, the other regular, G-stable subgroups of $\mathrm{Perm}(G)$ giving the same brace are of the form $N_\varphi,\ \varphi \in \mathrm{Aut}(G)$.

Outline

Equivalences

Let N_1, N_2 be regular, G-stable subgroups of $\mathrm{Perm}(G)$.

Recall N_1 and N_2 are G-isomorphic iff $L[N_1]^G \cong L[N_2]^G$ as K-Hopf algebras.

We say N_1 and N_2 are:

- *brace equivalent* if $\mathfrak{B}(N_1) \cong \mathfrak{B}(N_2)$
- *G-isomorphic* if there is a G-equivariant isomorphism $N_1 \to N_2$.

For either equivalence it is necessary, not sufficient, that $N_1 \cong N_2$ as groups.

Equivalences

- *brace equivalent*: $\mathfrak{B}(N_1) \cong \mathfrak{B}(N_2)$
- *G-isomorphic*: there is a G-equivariant isomorphism $N_1 \to N_2$.

Given $N \leq \mathrm{Perm}(G)$ regular and G-stable, let $\mathrm{BEC}(N)$ denote the brace equivalence class of regular, G-stable subgroups.

Then

$$\mathrm{BEC}(N) = \{N_\varphi : \varphi \in \mathrm{Aut}(G)\}/\sim,$$

where $N_{\varphi_1} \sim N_{\varphi_2}$ if $\varphi_1 = \varphi_2\varphi$ for some $\varphi \in \mathrm{Aut}(\mathfrak{B}(N)) \leq \mathrm{Aut}(G)$.

An example

Let $G = D_p = \langle r, s : r^p = s^2 = rsrs = 1_G \rangle$, p an odd prime.

[Byott] There are $p+2$ regular, G-stable subgroups of $\mathrm{Perm}(G)$:

- $\lambda(G) \cong D_p$ (left representation)
- $\rho(G) \cong D_p$ (right representation)
- $N_c = \langle \lambda(r)\rho(r^c s)\rangle \cong C_{2p}, \ 0 \leq c \leq p-1$.

$\mathfrak{B}(\lambda(G))$ is the trivial brace, so $\mathrm{Aut}(B, \cdot, \circ) = \mathrm{Aut}(B, \circ)$ and

$$\mathrm{BEC}(\lambda(G)) = \{\lambda(G)\}.$$

Since $\rho(G) \not\cong N_c$, $\mathrm{BEC}(\rho(G)) = \{\rho(G)\}$.

Pick c, and write $\mathfrak{B}(N_c) = (B, \cdot, \circ)$.

Since $\mathrm{Aut}(B, \cdot, \circ) \leq \mathrm{Aut}(B, \cdot) \cong C_{p-1}$ we know $|\,\mathrm{Aut}(B, \cdot, \circ)| \leq p-1$.

Since $|\,\mathrm{Aut}(B, \circ)| = p(p-1)$ we have

$$|\,\mathrm{BEC}(N_c)| \geq \frac{p(p-1)}{p-1} = p,$$

and since clearly $|\,\mathrm{BEC}(N_c)| \leq p$ we have equality, hence

$$\mathrm{BEC}(N_c) = \{N_0, N_1, \ldots, N_{p-1}\}$$

and we have three brace equivalence classes in total.

The three braces with $(B, \circ) \cong D_p$:

1. $(D_p, \cdot, \cdot)$
2. $(D_p, \cdot', \cdot)$ with $x \cdot' y = yx$
3. $(\langle \eta \rangle, \cdot, \circ)$, $|\eta| = 2p$ under $\cdot$, and

$$\eta^i \circ \eta^j = \eta^{i+(-1)^i j} = \begin{cases} \eta^{i+j} & i \text{ even} \\ \eta^{i-j} & i \text{ odd} \end{cases} .$$

$\{\lambda(G)\}, \{\rho(G)\}, \{N_c\}$

In this example, we also know the Hopf algebra isomorphism classes.

Let $H_\lambda = L[\lambda(G)]^G$ and $H_c = L[N_c]^G,\ 0 \leq c \leq p-1$.

Also, write $\eta_c = \lambda(r)\rho(r^c s)$ (so $N_c = \langle \eta_c \rangle$).

Then:

- $K[G]$ is in a class by itself.
- H_λ is in a class by itself.
- For all $0 \leq c, d \leq p-1$, $H_c \cong H_d$: since

$$
\begin{aligned}
{}^r\eta_c &= \lambda(r)\lambda(r)\rho(r^c s)\lambda(r^{-1}) = \lambda(r)\rho(r^c s) = \eta_c \\
{}^s\eta_c &= \lambda(s)\lambda(r)\rho(r^c s)\lambda(s^{-1}) = \lambda(r^{-1})\rho(r^c s) = \eta_c^{-1}
\end{aligned}
$$

  the map $N_c \to N_d,\ \eta_c \mapsto \eta_d$ is a G-equivariant isomorphism.

$\{\lambda(G)\}, \{\rho(G)\}, \{N_c\}$ and $H_\lambda, K[G], K[N_0]^G$

In this case, “brace equivalent” is the same thing as “G-isomorphic”.

Question. Is this always true?

Answer. No.

Example

Let $G = D_4$, let

$$\eta_r = \lambda(r)\rho(s),\ \eta_s = \lambda(s),$$

and let $N = \langle \eta_r, \eta_s \rangle$. Then the map $\theta : \lambda(G) \to N, \theta(\lambda(r^i s^j)) = \eta_r^i \eta_s^j$ is an isomorphism, and since

$$\begin{array}{cccc} {}^r\lambda(r) = \lambda(r) & {}^s\lambda(r) = \lambda(r)^{-1} & {}^r\lambda(s) = \lambda(r)^2\lambda(s) & {}^s\lambda(s) = \lambda(s) \\ {}^r\eta_r = \eta_r & {}^s\eta_r = \eta_r^{-1} & {}^r\eta_s = \eta_r^2\eta_s & {}^s\eta_s = \eta_s \end{array}$$

we see that θ is G-equivariant. Thus, $\lambda(G)$ is G-isomorphic to N. But $\mathrm{BEC}(\lambda(G)) = \{\lambda(G)\}$ so they are not brace equivalent.

Question. Does brace equivalence imply G-isomorphic?

Answer. Also, no.

Example

Let $G = \langle a, b : a^4 = abab^{-1} = 1, a^2 = b^2 \rangle \cong Q_8$, and let

$$N_1 = \langle \lambda(a), \lambda(ab)\rho(a) \rangle, \ N_2 = \langle \lambda(b), \lambda(ab)\rho(b) \rangle.$$

Both N_1 and N_2 are regular, G-stable, but not G-isomorphic since a acts trivially on N_1 and not N_2. [Taylor & Truman, 2019].
But $\varphi : G \to G, \ \varphi(a) = b, \ \varphi(b) = a$ is an automorphism, and

$$\varphi\lambda(a)\varphi^{-1}[a] = a^3 b \qquad \varphi\lambda(a)\varphi^{-1}[b] = a^2$$
$$\varphi\lambda(ab)\rho(a)\varphi^{-1}[a] = a^2 \qquad \varphi\lambda(ab)\rho(a)\varphi^{-1}[b] = a^3 b$$

Thus N_1 and N_2 are brace equivalent since $\varphi N_1 \varphi^{-1} = N_2$:

$$\varphi\lambda(a)\varphi^{-1} = (\lambda(b))^3 \in N_2, \ \varphi\lambda(ab)\rho(a)\varphi^{-1} = \lambda(b)^2(\lambda(ab)\rho(b)) \in N_2.$$

A special case

Let $\varphi \in \mathrm{Inn}(G)$, say $\varphi(g) = hgh^{-1}$. Let $\eta \in N$.

Then

$$\begin{aligned}\varphi\eta\varphi^{-1}[g] &= h(\eta[h^{-1}gh])h^{-1} \\ &= \rho(h)\lambda(h)\eta\lambda(h^{-1})\rho(h^{-1})[g].\end{aligned}$$

Since N is G-stable, ${}^h\eta = \lambda(h)\eta\lambda(h^{-1}) \in N$.

So $\varphi\eta\varphi^{-1} = \rho(h)\left({}^h\eta\right)\rho(h^{-1})$ and $N_\varphi = \rho(h)N\rho(h^{-1})$.

Since $\theta : N \to N_\varphi,\ \theta(\eta) = \rho(h)\eta\rho(h^{-1})$ is G-invariant [TARP19], we get that N and N_φ are G-isomorphic.

In particular:

Proposition (KT)

If $\mathrm{Aut}(G) = \mathrm{Inn}(G)$*, then brace equivalence implies isomorphic as Hopf algebras.*
Furthermore, if N_1 *and* N_2 *are in the same brace equivalence class, then*

$$N_2 = \rho(g) N_1 \rho(g^{-1})$$

for some $g \in G$*.*

An example: $(B, \circ) \cong (B, \cdot) = S_n, n \geq 5, n \neq 6$

There are two (opposite) families of regular, G-stable subgroups, denoted $\{N_\tau\}$ and $\{N'_\tau\}$, $\tau \in A_n$, $\tau^2 = 1$. Their $\circ$ brace operations are:

$$\sigma \circ \pi = \begin{cases} \sigma\pi & \sigma \in A_n \\ \sigma\tau\pi\tau & \sigma \notin A_n \end{cases}, \quad \sigma \circ' \pi = \begin{cases} \pi\sigma & \sigma \in A_n \\ \tau\pi\tau\sigma & \sigma \notin A_n \end{cases}.$$

Denote these braces $\mathfrak{B}_\tau$ and $\mathfrak{B}'_\tau$ respectively. Recall $\mathfrak{B}_\tau \not\cong \mathfrak{B}'_\tau$. Suppose $\tau_1, \tau_2 \in A_n$ are conjugate. Pick $\delta \in S_n$ such that $\delta\tau_1\delta^{-1} = \tau_2$. Let $\phi : \mathfrak{B}_{\tau_1} \to \mathfrak{B}_{\tau_2}$ be the bijection given by $\phi(\gamma) = \delta\gamma\delta^{-1}$. Then ϕ clearly preserves $\cdot$, and $\phi(\sigma \circ \pi) = \phi(\sigma) \circ \phi(\pi)$ for $\sigma \in A_n$. For $\sigma \notin A_n$,

$$\begin{aligned} \phi(\sigma \circ \pi) &= \phi(\sigma\tau_1\pi\tau_1) \\ &= \delta\sigma\tau_1\pi\tau_1\delta^{-1} \\ &= \delta\sigma(\delta^{-1}\delta)\tau_1(\delta^{-1}\delta)\pi(\delta^{-1}\delta)\tau_1\delta^{-1} \\ &= (\delta\sigma\delta^{-1})\tau_2(\delta\pi\delta^{-1})\tau_2 = \phi(\sigma) \circ \phi(\pi). \end{aligned}$$

$\mathfrak{B}_{\tau_1} \cong \mathfrak{B}_{\tau_2}$ if τ_1, τ_2 same cycle type: a special case

In particular, the brace equivalence classes for $G \cong N = S_7$ are:

$$\{\lambda(G)\}, \{\rho(G)\}, \{N_\tau : \tau^2 = 1, \tau \neq 1\}, \{N'_\tau : \tau^2 = 1, \tau \neq 1\},$$

and all Hopf algebras which give a Hopf-Galois structure on L/K are isomorphic to one of the following:

1. H_λ, which acts uniquely.
2. $K[G]$, which acts uniquely.
3. $K[N_{(12)(34)}]^G$, which acts in 105 ways.
4. $K[N'_{(12)(34)}]^G$, which acts in 105 ways.

G-isomorphism and brace equivalence: generally?

Proposition (KT)

Let $N_1, N_2 \leq \mathrm{Perm}(G)$ be regular and G-stable.
Suppose $\theta : N_1 \to N_2$ is G-equivariant, and $\varphi \in \mathrm{Aut}(G)$.
Then $\varphi\theta\varphi^{-1} : \varphi N_1 \varphi^{-1} \to \varphi N_2 \varphi^{-1}$ is G-equivariant.

So if N_1 and N_2 are G-isomorphic, then every element in $\mathrm{BEC}(N_1)$ is G-isomorphic to some element in $\mathrm{BEC}(N_2)$.

(This does not imply the equivalence classes are the same size.)

Back to the dihedral

Example

Let $G = D_4$, let

$$\eta_r = \lambda(r)\rho(s),\ \eta_s = \lambda(s),$$

and let $N = \langle \eta_r, \eta_s \rangle$.
We have seen that $\lambda(G)$ and N are G-isomorphic.
Since $\mathrm{BEC}(\lambda(G)) = \{\lambda(G)\}$, any group in $\mathrm{BEC}(N)$ must be G-isomorphic to $\lambda(G)$, hence to N, and the corresponding Hopf algebras are all isomorphic.

Generally, if a brace equivalence class has an element G-isomorphic to $\lambda(G)$, then all its elements are G-isomorphic.

Example: Braces Of Order $2p$

There are two possibilities for $G = \mathrm{Gal}(L/K)$, namely C_{2p} and D_p.

The case $G = D_p$ is handled earlier, giving three brace equivalence classes, corresponding to $\lambda(D_p),\ \rho(D_p),\ N_0 = \langle\lambda(r)\rho(s)\rangle$.

Suppose $G = C_{2p}$. Then N can be cyclic or dihedral.

If $N \cong C_{2p}$ then $N = \rho(C_{2p})$ and $\mathrm{BEC}(\rho(C_{2p})) = \{\rho(C_{2p})\}$.

If $N = \langle r, s\rangle \cong D_p$ then there are two Hopf-Galois structures [Byott], and they are opposite each other. One brace $\mathfrak{B} = (B, \cdot, \circ)$ is given by $(B, \cdot) = D_p$ and

$$r^i s^j \circ r^k s^l = r^{i+k} s^{j+l}.$$

If $\phi \in \mathrm{Aut}(B, \circ)$ then $\phi(r) = r^d$ and $\phi(s) = s$ for some $1 \leq d \leq p - 1$. These preserve the $\cdot$ operation as well, hence $\mathrm{Aut}(B, \cdot, \circ) = \mathrm{Aut}(G)$ and $\mathrm{BEC}(\mathfrak{B}) = \{\mathfrak{B}\}$.

Thus, there are 6 braces with $2p$ elements.

Braces of order $2p$

1. $(D_p, \cdot, \cdot)$
2. $(D_p, \cdot', \cdot)$ with $x \cdot' y = yx$.
3. $(\langle\eta\rangle, \cdot, \circ)$, $|\eta| = 2p$ under $\cdot$, and
$$\eta^i \circ \eta^j = \eta^{i+(-1)^i j} = \begin{cases} \eta^{i+j} & i \text{ even} \\ \eta^{i-j} & i \text{ odd} \end{cases} .$$
4. $(C_{2p}, \cdot, \cdot)$ (corresponding to $N = \rho(C_{2p})$).
5. $(D_p, \cdot, \circ)$ with $r^i s^j \circ r^k s^l = r^{i+k} s^{j+l}$.
6. $(D_p, \cdot', \circ)$ the opposite brace to the above.

Outline

Recall...

For $\mathfrak{B} = (B, \cdot, \circ)$ we define its opposite to be the brace $\mathfrak{B}' = (B, \cdot', \circ)$ with $x \cdot' y = yx$.

If $N' = \mathrm{Cent}_{\mathrm{Perm}(G)} N$, then $\mathfrak{B}(N') = \mathfrak{B}(N)'$.
Given N, we claim $|\,\mathrm{BEC}(N)| = |\,\mathrm{BEC}(N')|$.

In fact, for $\varphi \in \mathrm{Aut}(G), \eta \in N, \eta' \in N'$,

$$(\varphi\eta'\varphi^{-1})(\varphi\eta\varphi^{-1})(\varphi\eta'^{-1}\varphi^{-1}) = \varphi\eta'\eta\eta'^{-1}\varphi^{-1} = \varphi\eta\varphi^{-1}$$

and so:

Proposition (KT)

We have $(N_\varphi)' = (N')_\varphi$.

Generally, $\mathrm{BEC}(N_1) = \mathrm{BEC}(N_2)$ iff $\mathrm{BEC}(N_1') = \mathrm{BEC}(N_2')$.

In other words, each brace equivalence class has a well-defined opposite class (of the same size).

Self-opposite brace equivalence classes

Proposition (KT)

We have $(N_\varphi)' = (N')_\varphi$.

We have seen that it is possible for $\mathfrak{B} \cong \mathfrak{B}'$, i.e., for $\mathfrak{B}$ to be self-opposite.

Corollary

1. *If* $|BEC(N)|$ *is odd, then* $\mathfrak{B}(N)$ *is not self-opposite.*
2. *If* $|\mathrm{BEC}(M)| = |\mathrm{BEC}(N)|$ *implies* $\mathrm{BEC}(M) = \mathrm{BEC}(N)$, *then* $\mathfrak{B}(N)$ *is self-opposite.*

Outline

FPF abelian: review and notation

Recall that a *fixed-point free abelian* map $\psi : G \to G$ is an endomorphism such that $\psi(xy) = \psi(yx)$ for all $x, y \in G$; and $\psi(x) = x$ iff $x = 1_G$.

Let $\mathcal{F}(G)$ denote the set of fixed-point free abelian maps on G.

Each $\psi \in \mathcal{F}(G)$ determines a Hopf-Galois structure as follows. For $g \in G$, let

$$\eta_g^\psi = \eta_g := \lambda(g)\rho(\psi(g)) \in \mathrm{Perm}(G)$$

Let $N_\psi = \{\eta_g : g \in G\}$. Then $\eta_{gh} = \eta_g \eta_h$ and $N_\psi \leq \mathrm{Perm}(G)$.

Note $N_\psi \cong G$. It can be shown that N_ψ is regular and G-stable, hence gives a Hopf-Galois structure.

The correspondence $\mathcal{F}(G) \to \{\text{HGS on } L/K\}$ is neither injective nor surjective in general.

An injective correspondence

$\mathcal{F}(G) \to \{$ HGS on $L/K\}$

$\psi_1, \psi_2 \in \mathcal{F}(G)$ give the same HGS iff $\psi_2 = \psi_1\zeta$ for some $\zeta \in \mathcal{F}(G)$ such that $\zeta(G) \leq Z(G)$. Let $\mathcal{Z}(G)$ be the group of such maps.

We get an injective correspondence $\mathcal{F}(G)/\mathcal{Z}(G) \to \{$HGS on $L/K\}$. [Childs]

$\eta_g = \lambda(g)\rho(\psi(g))$

The map $\theta : \lambda(G) \to N_\psi$ given by $\theta(g) = \eta_g$ is an isomorphism, G-stable since

$$\begin{aligned} \theta({}^g\lambda(h)) &= \theta(\lambda(ghg^{-1})) = \eta_{ghg^{-1}}, \\ {}^g\theta(h) &= {}^g\eta_h \\ &= \lambda(g)\lambda(h)\rho(\psi(h))\lambda(g^{-1}) \\ &= \lambda(ghg^{-1})\rho(\psi(ghg^{-1})) \\ &= \eta_{ghg^{-1}}. \end{aligned}$$

Thus, every Hopf algebra arising from a fixed-point free map is necessarily isomorphic to H_λ. [Childs, TARP19]

Brace equivalence

Question. What does $\mathrm{BEC}(N_\psi)$ look like?

Short, unsatisfying answer. $\mathrm{BEC}(N_\psi) = \{\varphi N_\psi \varphi^{-1} : \varphi \in \mathrm{Aut}(G)\}$.

Longer, more satisfying answer. For $\varphi \in \mathrm{Aut}(G), \eta_g \in N_\psi$ we have

$$\begin{aligned}
\varphi\eta_g\varphi^{-1}[h] &= \varphi\left(\lambda(g)\rho(\psi(g))[\varphi^{-1}(h)]\right) \\
&= \varphi\left(g\varphi^{-1}(h)\psi(g^{-1})\right) \\
&= \varphi(g)h\varphi(\psi(g^{-1})) \\
&= \varphi(g)h\varphi(\psi(\varphi^{-1}(\varphi(g^{-1})))) \\
&= \lambda(\varphi(g))\rho(\varphi\psi\varphi^{-1}(\varphi(g)))[h].
\end{aligned}$$

$\varphi\eta_g\varphi^{-1} = \lambda(\varphi(g))\rho(\varphi\psi\varphi^{-1}(\varphi(g)))$

Claim. $\varphi\psi\varphi^{-1} \in \mathcal{F}(G)$.

$$\varphi\psi\varphi^{-1}(gh) = \varphi(\psi(\varphi^{-1}(g)\varphi^{-1}(h))) = \varphi(\psi(\varphi^{-1}(h)\varphi^{-1}(g))) = \varphi\psi\varphi^{-1}(hg)$$

$$\varphi\psi\varphi^{-1}(g) = g \Rightarrow \psi(\varphi^{-1}(g)) = \varphi^{-1}(g) \Rightarrow g = 1_G.$$

Thus, $\varphi\eta_g^{\psi}\varphi^{-1} = \eta_{\varphi(g)}^{\varphi\psi\varphi^{-1}}$ and

Proposition (KT)

The brace equivalence class for any N arising from a fixed-point free abelian map consists entirely of subgroups of Perm(G) *arising from fixed-point free abelian maps.*

Proposition (KT)

The brace equivalence class for any N arising from a fixed-point free abelian map consists entirely of subgroups of $\mathrm{Perm}(G)$ *arising from fixed-point free abelian maps.*

Recall that if N_{ψ_1} and N_{ψ_2} are both G-isomorphic (they are) and brace equivalent, $N_{\psi_2} = \rho(x)N_{\psi_1}\rho(x^{-1})$ for some $x \in G$.

Thus, N_{ψ_1} and N_{ψ_2} are brace equivalent iff there is an $x \in G$ such that for all $g \in G$ there is an $h \in G$ with

$$\rho(x)\lambda(g)\rho(\psi_1(g^{-1}))\rho(x^{-1}) = \lambda(g)\rho(x\psi_1(g^{-1})x^{-1}) = \lambda(h)\rho(\psi_2(h^{-1})).$$

Generally

Since the action of $\mathrm{Aut}(G)$ restricts to an action on $\mathcal{Z}(G)$:

$\mathrm{Aut}(G)$ acts on $\mathcal{F}(G)/\mathcal{Z}(G)$ via conjugation, and the orbits are brace equivalence classes.

An example

Let $D_8 = \langle r, s : r^8 = s^2 = rsrs = 1 \rangle$. There are 5 Hopf Galois structures given by fixed-point free maps:

	ψ_0	ψ_1	ψ_2	ψ_3	ψ_4
r	1_G	s	$r^2 s$	rs	$r^3 s$
s	1_G	1_G	s	rs	$r^3 s$
$\psi(G)$	$\{1_G\}$	$\langle s \rangle$	$\langle r^2, s \rangle$	$\langle rs \rangle$	$\langle r^3 s \rangle$
$\cong$ to	1	C_2	D_4	C_2	C_2

[Childs]

Write $N_i = N_{\psi_i}$.

Note $\varphi\psi\varphi^{-1}(G) = \varphi(\psi(G)) \cong \psi(G)$.

So $\mathrm{BEC}(N_0) = \{N_0\}$, $\mathrm{BEC}(N_2) = \{N_2\}$.

An example (cont'd)

	ψ_1	ψ_3	ψ_4
r	s	rs	r^3s
s	1_G	rs	r^3s
$\psi(G)$	$\langle s\rangle$	$\langle rs\rangle$	$\langle r^3s\rangle$

Let $\varphi \in \mathrm{Aut}(D_8)$ be given by $\varphi(r) = r,\ \varphi(s) = rs$. Then $\varphi\psi_1\varphi^{-1} = \psi_3$.

Also, the map $\varphi(r) = r^{-1},\ \varphi(s) = s$ satisfies $\varphi\psi_3\varphi^{-1} = \psi_4$, so

$$\mathrm{BEC}(N_1) = \{N_1, N_3, N_4\}.$$

Outline

A closer look at S_n

Earlier, we showed that, in the case $G \cong N = S_n,\ n \geq 5,\ n \neq 6$, HGS corresponding to the same choice $\tau \in A_n,\ \tau^2 = 1$ where $\sigma \circ \pi = \sigma\pi, \sigma \in A_n,\ \sigma \circ \pi = \sigma\tau\pi\tau, \sigma \notin A_n$ are all isomorphic.
(We explicitly looked at S_7.)

Is the converse true? If N_{τ_1} and N_{τ_2} are brace equivalent, must τ_1 and τ_2 be conjugate?

Conjecture. Yes (subject to the restrictions on n above).

If so, both the brace equivalence problem and the Hopf algebra isomorphism problem for $G \cong N = S_n$ would be completely known (modulo a few small, computable cases).

There is another class of regular, G-stable subgroup of $\mathrm{Perm}(G)$ with $N \cong A_n \times C_2$ which probably exhibits similar behavior.

A closer look at fixed-point free abelian

In the dihedral example we considered

	ψ_1	ψ_3	ψ_4
r	s	rs	r^3s
s	1_G	rs	r^3s
$\psi(G)$	$\langle s\rangle$	$\langle rs\rangle$	$\langle r^3s\rangle$

and showed that these were all are brace equivalent.

Is it true that, in general, if $\psi(G) \cong \chi(G)$ then N_ψ and N_χ are brace equivalent?

Conjecture. No.

The original problem

Our work in braces was motivated in determining when N and N' are G-isomorphic.

Is any of this helping?

Note we have shown that $(N_\varphi)' = N'_\varphi$, so N and N' are G-isomorphic iff N_φ and N'_φ are.

This separates the problem into “clusters”.

The original problem–an approach?

BEC N	BEC N'
N	N'
N_{φ_1}	N'_{φ_1}
N_{φ_2}	N'_{φ_2}
$\vdots$	$\vdots$
N_{φ_r}	N'_{φ_r}

N is G-isomorphic to N' iff N_{φ_i} is G-isomorphic to N'_{φ_i} for all i.

Perhaps there is a “nice" choice of N_{φ_i} for which it is obvious that N_{φ_i} and N'_{φ_i} are not G-isomorphic?

Thank you.