Fixed-Point-Free Abelian Endomorphisms, Braces, and the Yang-Baxter Equation

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Fixed-Point-Free Abelian Endomorphisms

FPF(G)

Definition

A fixed-point-free abelian endomorphism is a homomorphism $\psi : G \to G$ such that

- $\psi(g) = g$ if and only if $g = 1_G$ and
- $\psi(gh) = \psi(hg)$ for all $g, h \in G$.

We denote the collection of fixed-point-free abelian endomorphisms on a group G as FPF(G).

Example

Let $C_3 = \langle g : g^3 = 1_G \rangle$. Then $\psi_0(g) = 1_G$, $\psi_1(g) = g^2$ are fixed-point-free abelian endomorphisms.

Remark

If G is a nonabelian simple group, then FPF(G) consists only of the trivial ψ.

Since $\ker\psi \lhd G$, $\ker\psi = \{1_G\}$ or $\ker\psi = G$. The former causes a contradiction where $\psi(gh) \neq \psi(hg)$ for some pair g, h, thus $\ker\psi = G$.

Remark

ψ is constant on conjugacy classes, since

$$\psi(g) = \psi(ghh^{-1}) = \psi(ghg^{-1}).$$

Remark

Let $\psi \in \mathrm{FPF}(G)$, $\phi \in \mathrm{Aut}(G)$. Since $\phi\psi\phi^{-1} \in \mathrm{FPF}(G)$, FPF($G$) is stable under conjugation by Aut(G).

Regular, G-stable Subgroups

For $g \in G$, let $\eta_g = \lambda(g)\rho(\psi(g))$. Denote the collection of all such η_g's as N^{ψ}. We then have that N^{ψ} is a regular, G-stable subgroup of $\mathrm{Perm}(G)$.

Proposition (Childs)

If $\psi_1, \psi_2 \in \mathrm{FPF}(G)$ differ by an element of $Z(G)$, then $N^{\psi_1} \cong N^{\psi_2}$.

Example

Let $\psi_0 \in \mathrm{FPF}(G)$ denote the trivial fixed-point-free abelian endomorphism, i.e. $\psi_0(g) = 1_G$. Then $N^{\psi_0} = \lambda(G)$.

Braces

Braces

Definition

A left skew brace is a set B, along with two binary operations $\cdot$ and $\circ$, such that

- $(B, \cdot)$ and $(B, \circ)$ are groups, and
- For all $a, b, c \in B$, $a \cdot (b \circ c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$, where a^{-1} denotes the inverse of a in $(B, \cdot)$.

Denote the inverse of a in $(B, \circ)$ as $\bar{a}$.

Example

Let $(B, \cdot)$ be a group, and define $a \circ b = a \cdot b$. Then $(B, \cdot, \circ)$ is a brace.

Example

Let $B = \{0, 1, 2, 3, 4, 5\}$, and define $a \cdot b = a + b \bmod 6$ and $a \circ b = a + (-1)^a b \bmod 6$. Then $(B, \cdot, \circ)$ is a brace with $(B, \cdot) \cong C_6$ and $(B, \circ) \cong S_3$.

Regular Subgroups to Braces

Let N be a regular, G-stable subgroup of $\mathrm{Perm}(G)$ and let $a : N \to G$ be given by $a(\eta) = \eta[1_G]$.

Define $\eta \circ \pi = a^{-1}(a(\eta) \star_G a(\pi))$.

Proposition (Smoktunowicz-Vendramin)
$(N, \cdot, \circ)$ is a brace.

Denote this as $\mathfrak{B}(N)$.

Plugging N^{ψ} into this formula yields

$$\eta_g \circ \eta_h = \eta_{g\psi(g^{-1})h\psi(g)}.$$

Verify that this satisfies the brace condition. Note that ${\eta_g}^{-1} = \eta_{g^{-1}}$.

$$\begin{aligned}
\eta_g \circ (\eta_h \eta_k) &= \eta_g \circ \eta_{hk} \\
&= \eta_{g\psi(g^{-1})hk\psi(g)} \\
&= \eta_{g\psi(g^{-1})h\psi(g)g^{-1}g\psi(g^{-1})k\psi(g)} \\
&= \eta_{g\psi(g^{-1})h\psi(g)}\eta_{g^{-1}}\eta_{g\psi(g^{-1})k\psi(g)} \\
&= (\eta_g \circ \eta_h){\eta_g}^{-1}(\eta_g \circ \eta_k).
\end{aligned}$$

Brace Classes

Definition
If $\mathfrak{B}(N) \cong \mathfrak{B}(M)$, we say N and M are in the same brace class and call them brace equivalent.

Proposition (Koch-Truman)
$\mathfrak{B}(N^{\psi_1}) \cong \mathfrak{B}(N^{\psi_2})$ if and only if $\psi_1 = \phi\psi_2\phi^{-1}$.

The Yang-Baxter Equation

The Yang-Baxter Equation

Braces were constructed specifically with the objective of describing the set-theoretic solutions to the Yang-Baxter equation, functions $R : B \times B \to B \times B$, such that

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$$

where $R_{12}(a, b, c) = (R(a, b), c)$ and $R_{23}(a, b, c) = (a, R(b, c))$.

Given $\mathfrak{B} = (B, \cdot, \circ)$ we construct a solution $R : B \times B \to B \times B$.

Proposition (Guarnieri-Vendramin)
The following is a solution to the YBE:

$$R(a, b) = (a^{-1}(a \circ b), \overline{a^{-1}(a \circ b)} \circ a \circ b).$$

Example

The trivial brace gives us the solution $R(a,b) = (b, b^{-1}ab)$.

Verify this:

$$\begin{aligned} R_{12}R_{23}R_{12}(a,b,c) &= R_{12}R_{23}(b, b^{-1}ab, c) \\ &= R_{12}(b, c, c^{-1}b^{-1}abc) \\ &= (c, c^{-1}bc, c^{-1}b^{-1}abc) \end{aligned}$$

$$\begin{aligned} R_{23}R_{12}R_{23}(a,b,c) &= R_{23}R_{12}(a, c, c^{-1}bc) \\ &= R_{23}(c, cac^{-1}, c^{-1}bc) \\ &= (c, c^{-1}bc, c^{-1}b^{-1}abc). \end{aligned}$$

$R : N^\psi \times N^\psi \to N^\psi \times N^\psi$

We know the complete brace structure for $\mathfrak{B}(N^\psi)$, so we can construct the Yang-Baxter solution $R : N^\psi \times N^\psi \to N^\psi \times N^\psi$.

$$\begin{aligned} R(\eta_g, \eta_h) &= ({\eta_g}^{-1}(\eta_g \circ \eta_h), \overline{{\eta_g}^{-1}(\eta_g \circ \eta_h)} \circ \eta_g \circ \eta_h) \\ &= \left(\eta_{\psi(g^{-1})h\psi(g)}, \overline{\eta_{\psi(g^{-1})h\psi(g)}} \circ \eta_{g\psi(g^{-1})h\psi(g)}\right) \\ &= \left(\eta_{\psi(g^{-1})h\psi(g)}, \eta_{\psi(hg^{-1})h^{-1}\psi(gh^{-1})} \circ \eta_{g\psi(g^{-1})h\psi(g)}\right) \\ &= \left(\eta_{\psi(g^{-1})h\psi(g)}, \eta_{\psi(hg^{-1})h^{-1}\psi(g)g\psi(g^{-1})h\psi(gh^{-1})}\right). \end{aligned}$$

Constructions

Symmetric Group

Consider $S_n, n \neq 4, 6$.

The only normal subgroups in S_n are $\{\iota\}, A_n$, and S_n. We know $\ker \psi \neq \{\iota\}$, so for ψ to be a nontrivial fixed-point-free abelian endomorphism, $\ker \psi = A_n$.

Fix $\tau \in A_n$, $|\tau| = 2$.

Then $\psi_\tau \in \mathrm{FPF}(S_n)$, ψ_τ defined by

$$\psi_\tau(\pi) = \begin{cases} \iota \text{ if } \pi \in A_n \\ \tau \text{ if } \pi \notin A_n \end{cases}$$

We can check this: clearly $\{\iota, \tau\}$ is an abelian group, and $\psi_\tau(\tau) = \iota$, thus ψ_τ has no fixed points.

Let $\psi_1, \psi_2 \in \mathsf{FPF}(S_n)$, $\psi_1(S_n) = \langle \tau_1 \rangle$ and $\psi_2(S_n) = \langle \tau_2 \rangle$ where $\tau_2 = \pi \tau_1 \pi^{-1}$ for some $\pi \in S_n$.

Define $\phi \in \mathsf{Aut}(S_n)$ to be conjugation by π. Then $\psi_2 = \phi \psi_1 \phi^{-1}$, thus $\mathfrak{B}(N^{\psi_1}) \cong \mathfrak{B}(N^{\psi_2})$.

$R : S_n \times S_n \to S_n \times S_n$

Fix $\tau \in A_n$, $|\tau| = 2$. The solution set-theoretic solution to the Yang-Baxter equation arising from N^{ψ_τ} is

$$R(\eta_\pi, \eta_\chi) = \begin{cases} (\eta_{\tau\chi\tau}, \eta_{\chi^{-1}\tau\pi\tau\chi}) \text{ if } \pi, \chi \notin A_n \\ (\eta_{\tau\chi\tau}, \eta_{\tau\chi^{-1}\tau\pi\tau\chi\tau}) \text{ if } \pi \notin A_n,\ \chi \in A_n \\ (\eta_\chi, \eta_{\tau\chi^{-1}\pi\chi\tau}) \text{ if } \pi \in A_n,\ \chi \notin A_n \\ (\eta_\chi, \eta_{\chi^{-1}\pi\chi}) \text{ if } \pi, \chi \in A_n \end{cases}.$$

$R : S_n \times S_n \to S_n \times S_n$

We verify that this solution works in the case where $\pi, \chi, \sigma \notin A_n$:

$$
\begin{aligned}
R_{12}R_{23}R_{12}(\pi, \chi, \sigma) &= R_{12}R_{23}(\tau\chi\tau, \chi^{-1}\tau\pi\tau\chi, \sigma) \\
&= R_{12}(\tau\chi\tau, \tau\sigma\tau, \sigma^{-1}\tau\chi^{-1}\tau\pi\tau\chi\tau\sigma) \\
&= (\sigma, \tau\sigma^{-1}\tau\chi\tau\sigma\tau, \sigma^{-1}\tau\chi^{-1}\tau\pi\tau\chi\tau\sigma)
\end{aligned}
$$

$$
\begin{aligned}
R_{23}R_{12}R_{23}(\pi, \chi, \sigma) &= R_{23}R_{12}(\pi, \tau\sigma\tau, \sigma^{-1}\tau\chi\tau\sigma) \\
&= R_{23}(\sigma, \tau\sigma^{-1}\pi\sigma\tau, \sigma^{-1}\tau\chi\tau\sigma) \\
&= (\sigma, \tau\sigma^{-1}\tau\chi\tau\sigma\tau, \sigma^{-1}\tau\chi^{-1}\tau\pi\tau\chi\tau\sigma)
\end{aligned}
$$

Alternating Group

Let $A_4 = \langle \sigma, \upsilon \rangle$ with $\sigma = (123)$, $\upsilon = (124)$.We have four nontrivial fixed-point-free abelian endomorphisms on A_4:

1. $\psi_1(\sigma) = \sigma^2$, $\psi_1(\upsilon) = \sigma$
2. $\psi_2(\sigma) = \upsilon$, $\psi_2(\upsilon) = \upsilon^2$
3. $\psi_3(\sigma) = \upsilon^2\sigma$, $\psi_3(\upsilon) = \sigma^2\upsilon$
4. $\psi_4(\sigma) = \sigma\upsilon^2$, $\psi_4(\upsilon) = \upsilon\sigma^2$.

Alternating Group

The subgroups arising from the four nontrivial ψ's listed above are brace equivalent.

Let $\psi_a(\sigma) = \alpha, \psi_a(\upsilon) = \alpha^2$; $\psi_b(\sigma) = \beta, \psi_b(\upsilon) = \beta^2$. We have $\beta = \gamma\alpha\gamma^{-1}$ for some $\gamma \in S_4$.

Consider $\phi : A_4 \to A_4, \phi(\pi) = \gamma\pi\gamma^{-1}$ for all $\pi \in A_4$. Then $\phi\psi_a\phi^{-1} = \psi_b$, so $\mathfrak{B}(N^{\psi_1}) \cong \mathfrak{B}(N^{\psi_2})$.

$R : A_4 \times A_4 \to A_4 \times A_4$

Let $A_4 = \langle \sigma, \upsilon \rangle$, $\psi \in \mathrm{FPF}(A_4)$, $\psi(\sigma) = \alpha, \psi(\upsilon) = \alpha^2$.

Denote the conjugacy class of σ by $[\sigma]$, etc.

The values of $R(\eta_\pi, \eta_\chi)$ are given in the following table:

	$\chi \in [\sigma]$	$\chi \in [\upsilon]$	$\chi \in [\sigma\upsilon]$
$\pi \in [\sigma]$	$(\eta_{\alpha^2\chi\alpha}, \eta_{\chi^2\alpha\pi\alpha^2\chi})$	$(\eta_{\alpha^2\chi\alpha}, \eta_{\alpha\chi^2\alpha\pi\alpha^2\chi\alpha^2})$	$(\eta_{\alpha^2\chi\alpha}, \eta_{\alpha^2\chi\alpha\pi\alpha^2\chi\alpha})$
$\pi \in [\upsilon]$	$(\eta_{\alpha\chi\alpha^2}, \eta_{\alpha^2\chi^2\alpha^2\pi\alpha\chi\alpha})$	$(\eta_{\alpha\chi\alpha^2}, \eta_{\chi^2\alpha^2\pi\alpha\chi})$	$(\eta_{\alpha\chi\alpha^2}, \eta_{\alpha\chi\alpha^2\pi\alpha\chi\alpha^2})$
$\pi \in [\sigma\upsilon]$	$(\eta_\chi, \eta_{\alpha\chi^2\pi\chi\alpha^2})$	$(\eta_\chi, \eta_{\alpha^2\chi^2\pi\chi\alpha})$	$(\eta_\chi, \eta_{\chi\pi\chi})$

Note $R(\eta_\iota, \eta_\chi) = (\eta_\chi, \eta_\iota)$ and $R(\eta_\pi, \eta_\iota) = (\eta_\iota, \eta_\pi)$.

Metacyclic Group

Let $M_{pq} = \langle s, t : s^p = s^q = 1, ts = s^d t \rangle$ where d has order q in $\mathbb{Z}_p^\times$ and $p \equiv 1 \bmod q$.

The fixed-point free abelian endomorphisms on M_{pq} are of the form $\psi_{j,k} : M_{pq} \to M_{pq}$, $\psi_{j,k}(s) = 1, \psi_{i,j}(t) = s^j t^k$, with $k \neq 1,$ and $k = 0$ only if $j = 0$.

$\psi_{j,k}(s) = 1, \psi_{i,j}(t) = s^j t^k$

We can show that $\mathfrak{B}(N^{\psi_{1,k}}) \cong \mathfrak{B}(N^{\psi_{j,k}})$.

Let $\psi_{j,k} \in \mathrm{FPF}(M_{pq})$, $\psi_{j,k}(s) = 1$, $\psi_{j,k}(t) = s^j t^k$.

Case 1: $j \neq 0$.

Let $\phi \in \mathrm{Aut}(M_{pq})$, $\phi(s) = s^j, \phi(t) = t$.

Then $\phi\psi_{1,k} = \psi_{j,k}\phi$.

Case 2: $j = 0$.

Pick m such that

$$(1 + d + d^2 + \cdots + d^{k-1})m \equiv -1 \bmod p,$$

and define $\phi \in \mathrm{Aut}(G)$ by $\phi(s) = s, \phi(t) = s^m t$.

Then $\phi\psi_{1,k} = \psi_{0,k}\phi$.

Dihedral Group

Let $D_n = \langle r, s : r^n = s^2 = rsrs = 1 \rangle$.

Childs 2013 gives us FPF(D_n).

If n is odd, there are no nontrivial ψ's on D_n.

For n even, omitting the maps that differ by an element of the center, we have

1. $\psi(r) = 1, \psi(s) = 1$
2. $\psi(r) = r^i s, \psi(s) = 1$, i even
3. $\psi(r) = r^i s, \psi(s) = r^i s$, i odd.

Dihedral Group

All braces given by nontrivial ψ's are isomorphic.

1. $\psi(r) = r^i s, \psi(s) = 1$, i even
2. $\psi(r) = r^i s, \psi(s) = r^i s$, i odd.

Let $\psi_1 \in \mathrm{FPF}(D_{2m})$, $\psi_1(r) = r^2 s, \psi_1(s) = 1$.

Pick $\phi \in \mathrm{Aut}(D_{2m})$, $\phi(r) = r, \phi(s) = r^{i-2} s$.

Then $\phi\psi_1 = \psi\phi$ for all other ψ's.

$R(\eta_g, \eta_h) = \left(\eta_{\psi(g^{-1})h\psi(g)}, \eta_{\psi(hg^{-1})h^{-1}\psi(g)g\psi(g^{-1})h\psi(gh^{-1})}\right)$

1. $\psi(r) = r^i s, \psi(s) = 1$, i even: $\ker\psi = \langle r^2, s\rangle$
2. $\psi(r) = r^i s, \psi(s) = r^i s$, i odd: $\ker\psi = \langle r^2, rs\rangle$

$$R(\eta_g, \eta_h) = \begin{cases} (\eta_{r^i shr^i s}, \eta_{h^{-1}r^i sgr^i sh}) \text{ if } g, h \notin \ker\psi \\ (\eta_{r^i shr^i s}, \eta_{r^i sh^{-1}r^i sgr^i shr^i s}) \text{ if } g \notin \ker\psi,\ h \in \ker\psi \\ (\eta_h, \eta_{r^i sh^{-1}ghr^i s}) \text{ if } g \in \ker\psi,\ h \notin \ker\psi \\ (\eta_h, \eta_{h^{-1}gh}) \text{ if } g, h \in \ker\psi \end{cases}.$$

Thank you!