Skew Braces that do not come from Rota–Baxter Operators

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Omaha, 31 May 2022

10:30CDT 11:30EDT 15:30UTC 16:30BST 17:30CEST 00:30+1JST

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Skew braces and lambda/gamma functions

A skew brace is a set endowed with two group operations "$\cdot$" and "$\circ$", connected by

$$((a \cdot b) \circ c) \cdot c^{-1} = (a \circ c) \cdot c^{-1} \cdot (b \circ c) \cdot c^{-1}.$$

This states that the maps

$$\gamma(c) : x \mapsto (x \circ c) \cdot c^{-1}$$

are endomorphisms of $(G, \cdot)$.

In fact, a skew brace can be equivalently defined as a group $(G, \cdot)$, together with a map $\gamma : G \to \mathrm{Aut}(G)$ which satisfies the functional equation

$$\gamma(g^{\gamma(h)} h) = \gamma(g)\gamma(h).$$

This equation encodes the associativity of the group operation "$\circ$" given by $g \circ h = g^{\gamma(h)} h$.

Rota–Baxter operators: a short bibliography

Glen Baxter
An analytic problem whose solution follows from a simple algebraic identity
Pacific J. Math. **10** (1960), 731–742

Li Guo, Honglei Lang, and Yunhe Sheng
Integration and geometrization of R-B Lie algebras
Adv. Math. **387** (2021), Paper No. 107834, 34pp

Valeriy G. Bardakov and Vsevolod Gubarev
R–B groups, skew left braces, and the Y–B equation
J. Algebra **596** (2022), 328–351
https://arxiv.org/abs/2105.00428

A.C., L. Stefanello
Skew braces from R–B operators *etc*
arXiv:2201.03936, *Annali di Matematica Pura e Appl.*, 2022

Rota–Baxter operators and gamma functions

A Rota–Baxter operator on the group G is a map $B : G \to G$ such that

$$B(g^{B(h)}h) = B(B(h)^{-1}gB(h)h) = B(g)B(h).$$

Since gamma functions $\gamma : G \to \mathrm{Aut}(G)$ are characterised by the equation

$$\gamma(g^{\gamma(h)}h) = \gamma(g)\gamma(h),$$

if $\iota : G \to \mathrm{Inn}(G)$, $\iota : g \mapsto (x \mapsto g^{-1}xg)$, a Rota–Baxter operator B yields a gamma function

$$\gamma(g) = \iota(B(g)) \in \mathrm{Inn}(G), \tag{1}$$

via

$$G \xrightarrow{B} G \xrightarrow{\iota} \mathrm{Inn}(G), \qquad G \xrightarrow{\gamma} \mathrm{Inn}(G).$$

Conversely, if $\gamma : G \to \mathrm{Inn}(G)$ is a gamma function, under which conditions does it come from a Rota–Baxter operator via (1)?

Lifting Morphisms 1/3

Let U, V be groups, and A be an abelian, normal subgroup of V. Let $\varphi : U \to V/A$ be a morphism.

Under which conditions does φ lift to a morphism $U \to V$?

A is a V-module under conjugation, and thus a V/A-module, as A is abelian. A is then a U-module via φ.

Lift φ to a map $C : U \to V$, that is, for each $u \in U$ choose $C(u) \in \varphi(u)$, so that

$$\varphi(u) = C(u)A \quad \text{for } u \in U.$$

Since φ is a morphism, we will have

$$C(xy) = C(x)C(y)\kappa(x, y),$$

for some function $\kappa : U \times U \to A$.

Lifting Morphisms 2/3

$C : U \to V$ is a lift of the morphism $\varphi : U \to V/A$.

$$C(xy) = C(x)C(y)\kappa(x, y).$$

Let us enforce associativity:

$$\begin{aligned} C((xy)z) &= C(xy)C(z)\kappa(xy, z) \\ &= C(x)C(y)\kappa(x, y)C(z)\kappa(xy, z) \\ &= C(x)C(y)C(z)\kappa(x, y)^{z}\kappa(xy, z) \\ C(x(yz)) &= C(x)C(yz)\kappa(x, yz) \\ &= C(x)C(y)C(z)\kappa(y, z)\kappa(x, yz). \end{aligned}$$

We have obtained that κ is a 2-cocycle.

Lifting Morphisms 3/3

$C(xy) = C(x)C(y)\kappa(x, y)$, where $\kappa : U \times U \to A$ is a 2-cocycle.

- κ depends on the choice of the lift $C : U \to V$ of $\varphi : U \to V/A$, but
- the cohomology class of κ in $\mathrm{H}^2(U, A)$ is independent of C.

We have

Proposition

The following are equivalent.

- *$\varphi : U \to V/A$ lifts to a morphism $U \to V$.*
- *The class of κ in $\mathrm{H}^2(U, A)$ is trivial.*

A cohomological setting for Rota–Baxter operators

Let $\gamma : G \to \mathrm{Inn}(G)$ be a gamma function, so that $\gamma(g \circ h) = \gamma(g^{\gamma(h)} h) = \gamma(g)\gamma(h)$. Thus we have morphisms

$$\begin{array}{ccccc} (G, \circ) & \xrightarrow{\gamma} & \mathrm{Inn}(G) & \xrightarrow{\sim} & G/Z(G) \end{array}, \qquad \varphi : (G,\circ) \to G/Z(G)$$

Lift the morphism φ to a map $C : G \to G$ s.t. $\gamma(g) = \iota(C(g))$. Then

$$\begin{aligned} C(g^{C(h)} h) &= C(g^{\iota(C(h))} h) = C(g^{\gamma(h)} h) \\ &= C(g \circ h) = C(g)C(h)\kappa(g, h), \end{aligned}$$

where $\kappa : (G, \circ) \times (G, \circ) \to Z(G)$ is a 2-cocycle, which depends on the choice of C, but whose class in $\mathrm{H}^2((G, \circ), Z(G))$ does not.

Proposition

The following are equivalent.

- *γ comes from a Rota–Baxter operator B, i.e. $\gamma(g) = \iota(B(g))$.*
- *κ is trivial in $\mathrm{H}^2((G, \circ), Z(G))$.*

An example

Let $$(G, \cdot) = \langle\, u, v, k : u^p, v^p, k^p, [u, v] = k, [u, k], [v, k] \,\rangle.$$

be the Heisenberg group of order p^3, $p > 2$ a prime, $Z(G) = \langle\, k \,\rangle$.

Let $\alpha \in \mathbf{Z}/p\mathbf{Z}$. Consider the map $C : G \to G$ given by $C(g) = g^\alpha$. Then

$$\gamma(g) = \iota(g^\alpha) \text{ is a gamma function } G \to \mathrm{Inn}(G).$$

We have

- When $\alpha \neq -1/2$, the gamma function γ comes from the Rota–Baxter operator

  $$B(u^i \cdot v^j \cdot k^r) = u^{i\alpha} \cdot v^{j\alpha} \cdot k^{\alpha^2 (r - ij\alpha)(1+2\alpha)^{-1}},$$

  for $0 \leq i, j, r < p$.
- When $\alpha = -1/2$, the gamma function γ does *not* come from a Rota–Baxter operator. (Here $(G, \circ)$ is abelian.)

Skip Baer

Baer, Lazard, Baker–Campbell–Hausdorff

Reinhold Baer
Groups with abelian central quotient group
Trans. Amer. Math. Soc. **44** (1938), no. 3, 357–386

Let G be a group of nilpotence class two admitting unique square roots. Define

$$g \circ h = g \cdot h \cdot [g, h]^{-1/2}.$$

Then $(G, \circ)$ is an abelian group.

Skip calculation

$$\begin{aligned} h \circ g &= h \cdot g \cdot [h, g]^{-1/2} \\ &= g \cdot h \cdot [h, g] \cdot [h, g]^{-1/2} \\ &= g \cdot h \cdot [h, g]^{1/2} \\ &= g \circ h. \end{aligned}$$

Rota–Baxter operators via Extensions

The 2-cocycle associated to $C(g) = g^{\alpha}$ is $\kappa(x, y) = [x, y]^{-\binom{\alpha+1}{2}}$. Consider the standard sequence

$$1 \to Z(G) \to \underbrace{Z(G) \times (G, \circ)}_{\text{set-theoretic product}} \to (G, \circ) \to 1$$

associated to $\kappa \in \mathrm{H}^2((G, \circ), Z(G))$. The operation is given by

$$(z_1, g_1)(z_2, g_2) = (z_1 z_2 \kappa(g_1, g_2), g_1 \circ g_2).$$

- If the extension does not split, i.e. κ is non-trivial in $\mathrm{H}^2((G, \circ), Z(G))$, γ does not come from a R–B operator.
- If the extension does split, a complement to $Z(G)$ naturally determines a coboundary $\sigma : G \to Z(G)$, which is the correction to be made to C to obtain a R–B operator; recall

$$C(g^{C(h)} h) = C(g) C(h) \kappa(g, h).$$

How do we know whether it splits or not?

In the case of the sequence

$$1 \to Z(G) \to \underbrace{Z(G) \times (G, \circ)}_{\text{set-theoretic product}} \to (G, \circ) \to 1,$$

where $(G, \cdot) = \langle\, u, v, k : u^p, v^p, k^p, [u, v] = k, [u, k], [v, k], \,\rangle$.

one computes

$$[(1, u), (1, v)] = (k^{-\alpha(\alpha+1)}, k^{1+2\alpha}).$$

- If $\alpha = -1/2$, the sequence does not split; here $Z(G) \times 1$ is contained in the derived subgroup of the extension, so a complement to $Z(G) \times 1$ would be a maximal subgroup which does not contain the derived subgroup, a contradiction.
- If $\alpha \neq -1/2$, the subgroup $\langle\, (1, u), (1, v) \,\rangle$ intersects $Z(G) \times 1$ trivially, and thus it is a complement to $Z(G) \times 1$.
  The sequence splits (explicitly).

That's All, Thanks!