On Fuchs' problem on the group of units of a ring: the state of the art and some progress using braces

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Fuchs’ question

Fuchs' questions

In Fuchs' book "Abelian Groups" (1960) the following question is posed (Problem 72)

> Characterize the groups which are the (abelian) groups of all units in a commutative and associative ring with identity.

The general problem appeared to be very difficult and it is still open. Partial approaches

- to restrict the class of rings
- to restrict the class of groups
- to restrict both
- Ditor's question (1971). Which whole numbers can be the number of units of a ring?

The history of the problem

Units of number rings

Theorem (Dirichlet (1846))

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Let $[K : \mathbb{Q}] = r + 2s$ (here r is the number of real embeddings of K in $\bar{\mathbb{Q}}$ and $2s$ the number of non-real embeddings). Then

$$\mathcal{O}_K^* \cong T \times \mathbb{Z}^{r+s-1}$$

where T is the (cyclic) group of the roots of unity contained in K.

Units in group rings

Let R be a ring and let G be a group. The group ring RG is defined by

$$RG = \{\sum_{g \in G} \lambda_g g \mid \lambda_g \in R \text{ and } \lambda_g = 0 \text{ for almost all } g\}.$$

Theorem (Higman 1940) Let G be a finite abelian group of order n. Then

$$(\mathbb{Z}G)^* \cong \pm G \times \mathbb{Z}^{r_G}$$

where $r_G = \frac{1}{2}(n + 1 + c_2 - 2l)$, with
$c_d = \#\{\text{cyclic subgroups of order d of } G\}$ and $l = \sum_{d|n} c_n$.

More recently

- Pearson and Schneider (1970): Classification of the realizable cyclic groups.
- Chebolu and Lockridge (2015): Classification of the realizable indecomposable abelian groups.
- idc, Dvornicich (BLMS 2018 AMPA 2018)
  • Classification of the finite abelian groups realizable in the class of the integral domains/torsion-free rings/reduced rings.
  • For general rings $\rightarrow$ Necessary conditions for a finite ab. group to be realizable.
  $\rightarrow$ Infinite new families of realizable/non-real. finite abelian groups.
- idc (JLMS 2020) Classification of the finitely generated abelian groups which can be realized in the class of the integral domains, of the torsion-free rings and of the reduced rings.
- idc (work in progress) Some progress on classification using braces (radical rings).

Finitely generated abelian groups

Finitely generated abelian groups

Fuchs' question for finitely generated abelian groups

A ring with 1, A^* group of units of A. Assume that A^* is finitely generated and abelian

$$A^* \cong (A^*)_{tors} \times \mathbb{Z}^{r_A}$$

Problem: what groups arise?

- T finite abelian group: $\exists\ A \in \mathcal{C}$ such that $(A^*)_{tors} \cong T$?

- If $(A^*)_{tors} \cong T$ what can we say on $r_A = \mathrm{rank}(A^*)$?

Reduction step 1

Let $A_0 (= \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z})$ be the fundamental subring of A and consider the ring $R = A_0[(A^*)_{tors}]$.

Clearly $R^* \leq A^*$, therefore

$$r_A \geq r_R$$

and also

$$(A^*)_{tors} = (R^*)_{tors}.$$

So, up to changing $A \longleftrightarrow R = A_0[(A^*)_{tors}]$, we can restrict to study:

commutative rings which are finitely generated and integral over A_0.

Results for special classes of rings

Integral domains

Theorem (idc JLMS 2020)

The finitely generated abelian groups that occur as groups of units of an integral domain are:

i) $\mathrm{char}(A) = p$*: all groups of the form* $\mathbb{F}_{p^n}^* \times \mathbb{Z}^r$ *with* $n \geq 1$ *and* $r \geq 0$*;*
ii) $\mathrm{char}(A) = 0$*: all groups of the form* $\mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}^r$*, with* $n \geq 1$*,*
$r \geq \frac{\phi(2n)}{2} - 1$*.*

Corollary

The finite abelian groups that occur as groups of units of an integral domain A are:

i) the multiplicative groups of the finite fields if $\mathrm{char}(A) > 0$*;*

ii) the cyclic groups of order 2,4, or 6 if $\mathrm{char}(A) = 0$*.*

Torsion-free rings

A is *torsion-free* if 0 is the only element of finite additive order. In this case, $\mathrm{char}(A) = 0$.
Example: If R is a torsion-free ring and G is a group, then RG is torsion-free.

Theorem (idc JLMS 2020)

Let T be a finite abelian group of even order. Then there exists an explicit constant $g(T)$ such that the following holds:

$$T \times \mathbb{Z}^r$$

is the group of units of a torsion-free ring if and only if $r \geq g(T)$.

$$T \cong \prod_{\iota=1}^{s} \mathbb{Z}/p_\iota^{a_\iota}\mathbb{Z} \times \prod_{i=1}^{\rho} \mathbb{Z}/2^{\epsilon_i}\mathbb{Z} \times (\mathbb{Z}/2^{\epsilon}\mathbb{Z})^{\sigma}$$

where $s, \rho \geq 0$, $\sigma \geq 1$ and

- for all $\iota = 1, \ldots, s$ the p_ι's are odd prime numbers, not necessarily distinct, and $a_\iota \geq 1$;

- $\epsilon = \epsilon(T) \geq 1$ and $\epsilon_i > \epsilon$ for all $i = 1, \ldots, \rho$.

$$g(T) = \sum_{\iota=1}^{s}\left(\frac{\phi(2^{\epsilon} p_\iota^{a_\iota})}{2} - 1\right) + \sum_{i=1}^{\rho}\left(\frac{\phi(2^{\epsilon_i})}{2} - 1\right) + c(T)$$

where

$$c(T) = \begin{cases} (\sigma - s)\left(\frac{\phi(2^{\epsilon})}{2} - 1\right) & \text{for } s < \sigma \text{ and } \epsilon > 1 \\ 0 & \text{for } s_0 \leq \sigma \leq s \text{ or } \epsilon = 1 \\ \left\lceil \frac{\phi(2^{\epsilon})}{2} - 1 \right\rceil & \text{for } \sigma < s_0 \end{cases}$$

where $s_0 = \#\{p_1, \ldots, p_s\}$.

Corollary (idc, R.Dvornicich BLMS 2018)

The finite abelian groups which are the group of units of a torsion-free ring A, are all those of the form

$$(\mathbb{Z}/2\mathbb{Z})^a \times (\mathbb{Z}/4\mathbb{Z})^b \times (\mathbb{Z}/3\mathbb{Z})^c$$

where $a, b, c \in \mathbb{N}$, $a + b \geq 1$ *and* $a \geq 1$ *if* $c \geq 1$.

In particular, the possible values of $|A^*|$ *are the integers* $2^d 3^c$ *with* $d \geq 1$.

Reduced rings

Theorem (idc JLMS 2020)

The finitely generated abelian groups that occur as groups of units of a reduced ring are those of the form

$$\prod_{i=1}^{k} \mathbb{F}^*_{p_i^{n_i}} \times T \times \mathbb{Z}^g$$

where $k, n_1, \ldots, n_k$ are positive integers, $\{p_1, \ldots, p_k\}$ are, not necessarily distinct, primes, T is any finite abelian group of even order and $g \geq g(T)$.

Tools

An exact sequence

Let $\mathfrak{N} = \{a \in A \mid a^n = 0 \text{ for some } n \in \mathbb{N}\}$ be the nilradical of A. Clearly $1 + \mathfrak{N} < A^*$, so we have the following exact sequence:

$$1 \to 1 + \mathfrak{N} \to A^* \to A^*/(1 + \mathfrak{N}) \to 1$$

$$1 \to 1 + \mathfrak{N} \to A^* \to (A/\mathfrak{N})^* \to 1$$

The quotient ring $A/\mathfrak{N}$ is *reduced*, namely, its nilradical is trivial.

This exact sequence does not split in general.

Reduction step 2: splitting of the ring

Proposition (Pearson & Schneider 1970)

Let A be a commutative ring which is finitely generated and integral over its fundamental subring. Then $A = A_1 \oplus A_2$, where A_1 is a finite ring and the torsion ideal of A_2 is contained in its nilradical.

We will say that A is *of type 2* if its torsion ideal is contained in the nilradical.

If A is a type 2 ring $\Rightarrow \mathrm{char}(A) = 0$ since 1 is not nilpotent.

We can split the problem in the study of the units of finite rings and of characteristic 0 rings of type 2.

Finite rings

Finite rings

Let A be finite and commutative, then A is artinian and therefore $A \cong A_1 \times \cdots \times A_s$ (where A_i are *local* artinian rings) and

$$A^* \cong A_1^* \times \cdots \times A_s^*.$$

Let $(A, \mathfrak{m})$ be finite a local ring, and let $\mathrm{char}(A) = p^c$. In this case $\mathfrak{N} = \mathfrak{m}$ and the exact sequence is

$$1 \to 1 + \mathfrak{N} \hookrightarrow A^* \to (A/\mathfrak{N})^* \to 1$$

$$1 \to 1 + \mathfrak{m} \hookrightarrow A^* \to \mathbb{F}_{p^\lambda}^* \to 1$$

This sequence is split and therefore

$$A^* \cong (A/\mathfrak{m})^* \times (1 + \mathfrak{m}) \cong \mathbb{F}_{p^\lambda}^* \times (1 + \mathfrak{m}).$$

What can we say on the abelian p-group $1 + \mathfrak{m}$?

The abelian p-group $1 + \mathfrak{m}$.

- $|1 + \mathfrak{m}| = p^{k\lambda}$ for some $k \geq 0$
- Positive results. $\forall\, P$ abelian p-group there exists $(A, \mathfrak{m})$ with

$$A^* \cong \mathbb{F}_{p^\lambda}^* \times P^\lambda.$$

  In particular, all groups of type $\mathbb{F}_p^* \times P$ are the groups units of finite local rings. Here $p > 2$.

- On the negative side $\rightarrow 1 + \mathfrak{m}$ can be different from P^λ, but it can not be any p-group if $\lambda > 1$ and not even any group of cardinality $p^{k\lambda}$.

  Example: For $\lambda > 1$, the group $1 + \mathfrak{m}$ can not be cyclic.

  The presence of a "big" residue field gives an obstruction.

Characteristic zero rings

Characteristic 0 rings

We now restrict to the case when A^* is a **finite abelian** group.

Theorem (idc, R.Dvornicich)

If $\mathrm{char}(A) = 0$, *then*

$$A^* \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times H \\ \mathbb{Z}/4\mathbb{Z} \times H \end{cases},$$

where H *is finite and abelian.*

As a partial converse, we have that every group of type

$$\mathbb{Z}/2\mathbb{Z} \times H,$$

where H *is a finite abelian group, occur as group of units of a characteristic 0 ring.*

The last theorem, together with the result on finite rings, allows us to completely answer Ditor's question for rings of any characteristic.

Corollary

- The possible values of $|A^*|$, when A is a characteristic 0 ring with finite group of units, are all the even positive integers.
- The possible values of $|A^*|$, when A is a ring with finite group of units, are all the even positive integers and the finite products of integers of the form $2^\lambda - 1$ with $\lambda \geq 1$.

The case $\epsilon = 2$

We have seen that

$$A^* \cong \mathbb{Z}/2^\epsilon\mathbb{Z} \times H \text{ with } \epsilon = 1 \text{ or } 2$$

For $\epsilon = 1$ all groups H are possible.

The same is no longer true for rings with $\epsilon = 2$.

Example: We can not have $H \cong \mathbb{Z}/11\mathbb{Z}$, since the cyclic group $\mathbb{Z}/44\mathbb{Z}$ is not realizable.

$$\text{If } \epsilon(A) = 2, \text{ then } \mathbb{Z}[i] \subseteq A.$$

The presence of the ring $\mathbb{Z}[i]$ is the obstruction in this case.

Consider the exact sequence

$$1 \to 1 + \mathfrak{N} \to A^* \to (A/\mathfrak{N})^* \to 1$$

and the exact sequence induced on the p-Sylow

$$1 \to (1 + \mathfrak{N})_p \to (A^*)_p \to (A/\mathfrak{N})_p^* \to 1.$$

This can be rewritten as

$$1 \to (1 + \mathfrak{N}_p) \to (A^*)_p \to B_p^* \to 1$$

where $B = A/\mathfrak{N}$.

If **A** is a **type 2 ring**, then the ring B is torsion-free and B^* is described by our classification. In particular,

$(B^*)_p$ is trivial for $p > 3$ and also for $p = 3$ if $\epsilon(A) = 2$.

Hence

$$(A^*)_p = 1 + \mathfrak{N}_p \ \forall p \geq 3.$$

In the case when A is of type 2 with $\epsilon(A) = 2$ we have found some necessary and some sufficient conditions, which are indeed very strict but not conclusive.

- $p \equiv 1 \pmod 4$: $(A^*)_p = 1 + \mathfrak{N}_p$ can be any abelian p-group.
- $p \equiv 3 \pmod 4$:
  - $(A^*)_p = 1 + \mathfrak{N}_p$
  - it can not be any p-group (e.g. it can not be cyclic),
  - the cardinality of $(A^*)_p$ must be a square,
  - and all squares of a p-group are realizable.
- $p = 2$: we have an exact sequence

$$1 \to 1 + \mathfrak{N}_2 \to (A^*)_2 \to (\mathbb{Z}/2\mathbb{Z})^a \times (\mathbb{Z}/4\mathbb{Z})^b \to 1$$

  where $a + b \geq 1$. We have a (not exhaustive) list of realizable 2-Sylow subgroups of A^*.

The radical ring $\mathfrak{N}$

The radical ring $\mathfrak{N}$

Both in the case $char(A) > 0$ and in the case $char(A) = 0$, we are left to study the group

$$1 + \mathfrak{N}$$

• When $char(A) > 0$ and A local $\implies A^* \cong \mathbb{F}^*_{p^\lambda} \times 1 + \mathfrak{N}$
the knowledge of $1 + \mathfrak{N}$ would be enough the conclude the characterization of the groups of units arising in this case.

• When $char(A) = 0 \implies A^*_p = 1 + \mathfrak{N}_p$, for $p > 2$
(the knowledge of $1 + \mathfrak{N}_2$ is not sufficient to determine the 2-Sylow of A^*.)

$\mathfrak{N}$ is a radical ring, so we can consider on it the adjoint operation $\circ$ defined by

$$x \circ y = x + y + xy, \qquad \forall x, y \in \mathfrak{N}$$

we have that $(\mathfrak{N}, +, \circ)$ is a (two-sided) brace and $1 + \mathfrak{N} \cong (\mathfrak{N}, \circ)$.

The following theorem gives some relation between the two group structures of a brace.
(For a p-group G we call $\mathrm{rank}(G)$ is the maximum r such that G has a subgroup of exponent p and order p^r. If G is abelian it is the number of cyclic factors of its decomposition as a product of cyclic groups)

Theorem (FCC12 - Bac16 - Caranti idc 22)

Let p be a prime, and let $(G, +, \circ)$ be a brace of order a power of the prime p. Then

$$rank(G, +) < p - 1 \iff rank(G, \circ) < p - 1.$$

When these conditions hold, $(G, +)$ and $(G, \circ)$ have the same rank, and each element has the same order in $(G, +)$ and $(G, \circ)$.

[FCC12] $\Longrightarrow$ if $\mathrm{rank}(\mathfrak{N}_p) < p-1$ then $1+\mathfrak{N}_p \cong \mathfrak{N}_p$

Q1 What kind of abelian group can be $\mathfrak{N}_p$?

Q2 Can we weaken the condition on the rank in [FCC12] and get the same result for $1+\mathfrak{N}_p$?

Q1: we are looking for some *general* information on $\mathfrak{N}p$.

Q2 seems to be hopeless: [FCC12] has been refined by Bachiller and by Caranti and myself, but well known examples show that it can not be generalized to braces (or radical rings) of rank$= p-1$, so we have to understand what kind of generalization is possible.

Q1

Consider the case $(A, \mathfrak{m})$ local with $A/\mathfrak{m} \cong \mathbb{F}^*_{p^\lambda}$, but everything has an analog for characteristic 0 rings!

In this case $\mathfrak{N} = \mathfrak{N}_p = \mathfrak{m}$.

$\mathfrak{m}$ can be the λ power of any abelian p-group.

In fact, let $P \cong \mathbb{Z}/p^{a_0}\mathbb{Z} \times \mathbb{Z}/p^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{a_r}\mathbb{Z}$ $(a_0 \geq a_i)$.

Define $R = \frac{(\mathbb{Z}/p^{a_0+1}\mathbb{Z})[t]}{(f(t))}$ where $f(t)$ is irreducible modulo p of degree λ. Then

$$A = \frac{R[x_1, \dots x_r]}{(p^{a_i}x_i, x_i x_j)_{1 \leq i,j \leq r}}$$

is such that $\mathfrak{m} \cong P^\lambda$.

Q1

The following Theorem shows that also the converse holds, i.e.

$\mathfrak{m}$ is always the λ power of an abelian p-group.

Theorem

$(A, \mathfrak{m})$ local, with $A/\mathfrak{m} \cong \mathbb{F}_{p^\lambda}$. Then,

$$\mathfrak{m} \cong P^\lambda$$

where P is an abelian p-group.

Sketch of the proof. Let $D_\lambda = \mathbb{Z}_p[\zeta_{p^\lambda - 1}]$, where $\zeta_{p^\lambda - 1} \in \bar{\mathbb{Q}}_p$ is a root of unit of order $p^\lambda - 1$ (notice that D_λ is the ring of integers of the unramified extension of $\mathbb{Q}_p$ of degree λ).

We can prove that $\mathfrak{m}$ is a D_λ-module. Being D_λ a DVR, we have

$$\mathfrak{m} \cong \bigoplus_{i=1}^{n} D_\lambda / p^{a_i} D_\lambda \cong (\bigoplus_{i=1}^{n} \mathbb{Z}/p^{a_i}\mathbb{Z})^\lambda,$$

where the first is an isomorphism of D_λ-modules and the second as groups. □

Generalizing FCC12 and Bac15

$\mathfrak{m} = P^\lambda$: [FCC12] applies only when $\text{rank} P^\lambda = \lambda \text{rank} P < p - 1$.

For $\lambda \geq p - 1$ [FCC12] gives NO information on $1 + \mathfrak{m}$.

Theorem (idc)

Let p be a prime number, and let D be a PID such that p is a prime in D.

Let $(N, +)$ is a D-module of order a power of p, with $\mathrm{rank_D N < p - 1}$,

Then if $(N, +, \circ)$ is a "brace" then each element has the same order in $(N, +)$ and $(N, \circ)$.

In particular, if $(N, \circ)$ is abelian, then $(N, +) \cong (N, \circ)$.

Here $\mathrm{rank_D N} = \#$summand of the dec. of N as a sum of cyclic $D-$mod.

[Bac15] gives the same result of last theorem for $D = \mathbb{Z}$.
Further examples: $D =$ the ring of integers of unramified extensions of $\mathbb{Q}_p$.
The theorem above admits a partial generalization to the case when D is a generic Dedekind domain, without any assumption on the factorization of pD. Here with "brace" we intend a two sided (+ some other) braces.

Generalization to Dedekind domanis

Theorem (idc)

Let $(N, +, \circ)$ be a two-sided (or also more general) brace of cardinality the power of a prime p.

Assume that $(N, +)$ is a $\mathcal{O}$-module, where $\mathcal{O}$ is the ring of integers of a number field or a p-adic field.

Let $p\mathcal{O} = P_1^{e_1} \cdots P_r^{e_r}$ be the factorization of $p\mathcal{O}$. For each $i = 1, \ldots, r$, let $f_i = [\frac{\mathcal{O}}{P_i} : \frac{\mathbb{Z}}{p\mathbb{Z}}]$ and denote by N_i the P_i-component of N.

If the $\mathbb{Z}$-rank of the abelian group N_i is $< f_i(p-1)$, for all $i = 1, \ldots, r$, then $(N, +)$ and $(N, \circ)$ have the same number of element of each order. In particular, if $(N, \circ)$ is abelian, then $(N, +) \cong (N, \circ)$.

The case of small rank: char(A)>0

Theorem (idc)

Let $(A, \mathfrak{m})$ be a finite local ring with residue field of cardinality p^λ.

If rank $\mathfrak{m} < \lambda(p-1)$, then

$$A^* \cong \mathbb{F}_{p^\lambda}^* \times P^\lambda$$

where P is an abelian p-group.

Sketch of the proof. The condition $\lambda(p-1) > \text{rank}\ \mathfrak{m} = \lambda \text{rank}_{D_\lambda} \mathfrak{m}$ gives

$$\text{rank}_{D_\lambda} \mathfrak{m} < p - 1$$

therefore the previous theorem applies giving $1 + \mathfrak{m} \cong \mathfrak{m}$. Since $\mathfrak{m} = P^\lambda$ for some P, we can conclude. □

We have a classification of all small finite abelian groups occurring as group of units of a finite ring.

The case of small rank: char(A)=0

Let A a type 2 ring with $\epsilon = 2$.

Then, $\mathfrak{N}$ is a $\mathbb{Z}[i]$-module.

In this case we say that an abelian group H is *small* if

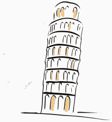
$$\mathrm{rank}(H_p) < 2(p-1), \text{ for all } p \equiv 3 \pmod 4.$$

Theorem (idc)

Let H be a small abelian group of odd order.

The group $\mathbb{Z}/4\mathbb{Z} \times H$, is the group of units of type 2 ring, if and only if H_p is the square of a group, $\forall p \equiv 3 \pmod 4$.

Thank you!



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