Commutator-central maps, brace blocks, and Hopf-Galois extensions

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Outline

Setup

Let $G = (G, \cdot)$ be a finite, nonabelian group, center Z and commutator subgroup $[G, G]$.

Denote by $\mathrm{Ab}(G)$ the set of endomorphisms $\psi : G \to G$ with $\psi(G)$ abelian.

Recall [K., 2021] any $\psi \in \mathrm{Ab}(G)$ gives a regular, G-stable subgroup $N := \{\eta_g : g \in G\}$ of $\mathrm{Perm}(G)$, where

$$\eta_g[h] = g\psi(g^{-1})h\psi(g).$$

Regular, G-stable subgroups $N \leq \mathrm{Perm}(G)$ give

- skew left braces;
- solutions to the Yang-Baxter equation; and
- Hopf-Galois structures on a G-extension of fields, and the *type* of the structure is the abstract group isomorphic to N.

Regular, G-stable subgroups give braces

A *skew left brace* (hereafter, *brace*) is a triple $(B, \cdot, \circ)$ where $(B, \cdot)$ and $(B, \circ)$ are groups and

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$$

holds for all $a, b, c \in B$, where $a \cdot a^{-1} = 1_B$. Childs denotes this $(B, \circ, \cdot)$.

The two simplest examples:

Example

For $(G, \cdot)$ any group, the triple $(G, \cdot, \cdot)$ is a brace. We call this the *trivial brace* on G.

Example

For $(G, \cdot)$ any nonabelian group, and define $g \cdot' h = hg$ for all $g, h \in G$. Then the triple $(G, \cdot, \cdot')$ forms the *almost trivial brace* on G.

Regular, G-stable subgroups give braces

A *skew left brace* (hereafter, *brace*) is a triple $(B, \cdot, \circ)$ where $(B, \cdot)$ and $(B, \circ)$ are groups and

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$$

holds for all $a, b, c \in B$, where $a \cdot a^{-1} = 1_B$. Childs denotes this $(B, \circ, \cdot)$.

Properties and Conventions

- $(B, \cdot)$ and $(B, \circ)$ have the same identity 1_B.
- We write the inverse to $a \in (B, \circ)$ by $\overline{a}$.
- We will frequently write $a \cdot b$ as ab.

Example (K, 2021)

Let $\psi \in \mathrm{Ab}(G)$, and define

$$g \circ h = \eta_g[h] = g\psi(g^{-1})h\psi(g).$$

Then $(G, \cdot, \circ)$ is a brace.

A conceptual break from earlier works

There is a well-known connection between regular, G-stable subgroups N of $\mathrm{Perm}(G)$ and braces. If $\varkappa : N \to G$ is given by $\varkappa(\eta) = \eta[1_G]$ then one defines an operation $\circ$ on N via:

$$\eta \circ \pi = \varkappa^{-1}(\varkappa(\eta) *_G \varkappa(\pi)).$$

One then has a brace $(N, \cdot, \circ)$ with $(N, \cdot) \leq \mathrm{Perm}(G, \circ)$.

That's not what's happening in our construction.

$$g \circ h = \eta_g[h] = g\psi(g^{-1})h\psi(g).$$

Our brace is $(G, \cdot, \circ)$ with $(G, \circ) \leq \mathrm{Perm}(G, \cdot)$.

This works because both $(G, \cdot, \circ)$ and $(G, \circ, \cdot)$ are braces (i.e., $(G, \cdot, \circ)$ is a *bi-skew brace*).

Braces give solutions to the Yang-Baxter equation

A *set-theoretic solution to the Yang-Baxter equation* (hereafter, *solution to the YBE*) is a set B and a map $R : B \times B \to B \times B$ such that

$$(R \times \mathrm{id}_B)(\mathrm{id}_B \times R)(R \times \mathrm{id}_B) = (\mathrm{id}_B \times R)(R \times \mathrm{id}_B)(\mathrm{id}_B \times R) : B^3 \to B^3.$$

A solution $R(x, y) = (\sigma_x(y), \tau_y(x))$ is *non-degenerate* if σ_x and τ_y are bijections, *involutive* if $R^2 = \mathrm{id}_{B\times B}$.

Generally, a brace $(B, \cdot, \circ)$ gives non-degenerate solutions:

$$R(a, b) = (a^{-1}(a \circ b), \overline{a^{-1}(a \circ b)} \circ a \circ b)$$
$$R^{-1}(a, b) = ((a \circ b)a^{-1}, \overline{(a \circ b)a^{-1}} \circ a \circ b).$$

Note that R is involutive if and only if $(B, \cdot)$ is abelian.

Abelian maps and solutions

$$R(a,b) = (a^{-1}(a \circ b), \overline{a^{-1}(a \circ b)} \circ a \circ b)$$
$$R^{-1}(a,b) = ((a \circ b)a^{-1}, \overline{(a \circ b)a^{-1}} \circ a \circ b).$$

Example (K., 2021)

For $\psi \in \mathrm{Ab}(G)$ we get the following solutions with underlying set G:

$$R(g,h) = (\psi(g^{-1})h\psi(g), \psi(hg^{-1})h^{-1}\psi(g)g\psi(g^{-1})h\psi(gh^{-1}))$$
$$R^{-1}(g,h) = (g\psi(g^{-1})h\psi(g)g^{-1}, \psi(h)g\psi(h^{-1}))$$

Note. There are two more solutions because $(G, \cdot, \circ)$ is a bi-skew brace, but we will not directly address the other two here.

More maps

Denote by $\mathsf{Map}(G)$ the set of all functions on G.

With the binary operations

$$(\alpha + \beta)(g) = \alpha(g)\beta(g), \ \alpha\beta(g) = \alpha(\beta(g)), \ g \in G$$

we have a right near-ring structure on $\mathsf{Map}(G)$, i.e.,

- $(\mathsf{Map}(G), +)$ is a (nonabelian) group;
- "multiplication" is associative; and
- $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$ for all $\alpha, \beta, \gamma \in \mathsf{Map}(G)$.

For $n \in \mathbb{Z}$ we write $n \in \mathsf{Map}(G)$ to represent $g \mapsto g^n$.

So $0, 1 \in \mathsf{Map}(G)$ are the trivial and identity map respectively.

Note that both $\mathsf{Ab}(G)$ and $\mathsf{End}(G)$ are contained in $\mathsf{Map}(G)$ but are not subgroups.

Last year's Omaha construction

Let $\psi \in \mathrm{Ab}(G)$.

Define $\{\psi_n : n \geq 0\}$ by $\psi_n = -(1-\psi)^n + 1$.

For example, $\psi_0 = 0,\ \psi_1 = \psi,\ \psi_2 = 2\psi - \psi^2$.

Then $\psi_n \in \mathrm{Ab}(G)$ for n.

Theorem (K., 2022)

Let $n \geq 0$ and $g \circ_n h = g\psi_n(g^{-1})h\psi_n(g)$. Then $(G, \cdot, \circ_n)$ is a brace. Furthermore, for all $m \geq 0$, $(G, \circ_m, \circ_n)$ is a brace.

We say G, together with $\{\circ_n : n \geq 0\}$, form a *brace block*.

Brace blocks

A *brace block* is a set B and a family $\{\circ_n : n \in \mathcal{I}\}$, $\mathcal{I}$ an index set such that $(B, \circ_m, \circ_n)$ is a brace for all $m, n \in \mathcal{I}$.

We will denote this brace block by $(B, \{\circ_n : n \in \mathcal{I}\})$

Such braces are necessarily bi-skew.

Short examples:

- $(G, \{\cdot\})$ is the trivial brace block.
- If $(G, \cdot, \circ)$ is a bi-skew brace, then $(G, \{\cdot, \circ\})$ is a brace block.
- If $\psi \in \mathrm{Ab}(G)$ then $(G, \{\circ_n : n \geq 0\})$ is a brace block.

Generalizations: C-S 2021 v. 1

The work on abelian maps and brace blocks is generalized in [Caranti-Stefanello 2021, v. 1].

The condition $\psi \in \mathrm{Ab}(G)$ can be relaxed: one can, for example, take $\psi \in \mathrm{End}(G)$ such that $\psi([G,G]) \leq Z(G)$.

We call such maps *commutator-central* and denote the set of all commutator central maps by $\mathtt{CC}(G)$.

Additionally, [C-S 21 v. 1] replaces $\psi_n = -(1-\psi)^n + 1$ with $\psi_n \in \psi\mathbb{Z}[\psi] \subset \mathtt{Map}(G)$ and creates a brace block with binary operations given recursively by

$$g \circ_n h = g \circ_{n-1} \psi_n(g) \circ_{n-1} h \circ_{n-1} \widetilde{\psi_n(g)},$$

where $g \circ_{n-1} \widetilde{g} = 1_G$.

Generalizations: B-N-Y 2022

Bardakov, Neshchadim, and Yadav talk about *brace systems*: a set G and a graph (V, E) where the vertices are binary operations and directed edges $\cdot \to \circ$ give braces $(G, \cdot, \circ)$.

A double-arrow corresponds to a “symmetric brace”, i.e., bi-skew brace.

They use “λ-homomorphisms” to construct brace blocks, which encompasses [K, 2022] and [C-S 21 v. 1].

These are also constructed recursively: $a \circ_{i+1} b = a \circ_i \lambda_a(b)$ where $\lambda_a : G \to \mathrm{Aut}(G)$ satisfies certain properties.

Motivation for current work

$$g \circ_n h = g \circ_{n-1} \psi_n(g) \circ_{n-1} h \circ_{n-1} \widetilde{\psi_n(g)}$$

$$a \circ_{i+1} b = a \circ_i \lambda_a(b)$$

Thoughts on seeing this construction

- Given the lack of "natural ordering" in the ψ_n's, the recursive nature to these definitions seems "artificial".
- It would be nice to write the binary operations non-recursively.
- A priori, there seems to be no reason why a brace block needs to be constructed as a sequence.
- The jump from my prescribed family of maps $\psi_n = -(1-\psi)^n + 1$ to the family in [C-S, 2021, v.1] or [B-N-Y 2022] is a significant one. Can we generalize even more?

Outline

The main construction

Throughout, fix $\psi \in \mathrm{CC}(G)$.

Note that $\psi \in \mathrm{CC}(G)$ means that $\psi(gh) = \psi(hg)z$ for some $z \in Z$.

Let $\mathscr{E}$ be the elements of $\mathrm{Map}(G)$ which decompose as a sum of endomorphisms.

So $\mathscr{E} = \{\alpha : \alpha = \phi_1 + \phi_2 + \cdots + \phi_n,\ \phi_i \in \mathrm{End}(G)\} \subset \mathrm{Map}(G)$.

Note $\mathrm{End}(G) \subsetneq \mathscr{E} \subsetneq \mathrm{Map}(G)$ since $-1 : g \mapsto g^{-1} \notin \mathrm{End}(G)$ and, e.g., $\alpha(1_G) = 1_G$ for all $\alpha \in \mathscr{E}$.

Let $\psi_\alpha = \psi\alpha$, and define

$$g \circ_\alpha h = g\psi_\alpha(g)h\psi_\alpha(g)^{-1}.$$

$g \circ_\alpha h = g\psi_\alpha(g)h\psi_\alpha(g)^{-1}$

Special cases:

$$\begin{aligned} \alpha = 0 &: \quad g \circ_\alpha h = gh \\ \alpha = -1 &: \quad g \circ_\alpha h = g\psi(g)^{-1}h\psi(g) = g \circ h && \text{[K, 2021]} \\ \alpha = 1 &: \quad g \circ_\alpha h = g\psi(g)h\psi(g)^{-1} && \text{[C-S 21 v. 1]} \\ \alpha = \sum_{i=0}^{n-1}(-1)^i\binom{n}{i}\psi^i &: \quad g \circ_\alpha h = g \circ_n h && \text{[K, 2022]} \end{aligned}$$

We also get the C-S 21 v. 1 blocks obtained from elements of $\psi\mathbb{Z}[\psi]$.

Theorem (K, c. 2023)

Let $\psi \in CC(G)$, $\alpha, \beta \in \mathscr{E}$. Then $(G, \circ_\alpha, \circ_\beta)$ is a brace.
In other words,

$$(G, \{\circ_\alpha : \alpha \in \mathscr{E}\})$$

is a brace block.

More observations:

- $g \circ_\alpha h = g \circ_\beta h$ for all $g, h \in G$ if and only if $\psi(\alpha - \beta)(G) \leq Z$.
- If α and β consist of the same endomorphisms, used the same number of times, then $g \circ_\alpha h = g \circ_\beta h$ for all $g, h \in G$.
  For example,

$$\begin{aligned}\psi((\phi_1 + \phi_2) - (\phi_2 + \phi_1))(g) &= \psi\left(\phi_1(g)\phi_2(g)(\phi_2(g)\phi_1(g))^{-1}\right) \\ &= \psi\left(\phi_1(g)\phi_2(g)\phi_1(g)^{-1}\phi_2(g)^{-1}\right) \in Z\end{aligned}$$

  So the ordering of the endomorphisms in an element of $\mathscr{E}$ doesn't matter: we can think of $\mathscr{E}$ as the free abelian group generated by $\mathrm{End}(G)$.

Back to the YBE

Each brace $(G, \circ_\alpha, \circ_\beta)$ in a brace block gives (potentially) two solutions to the YBE:

$$R(g,h) = (\widetilde{g} \circ_\alpha (g \circ_\beta h), \overline{\widetilde{g} \circ_\alpha (g \circ_\beta h)} \circ_\beta g \circ_\beta h)$$
$$R^{-1}(g,h) = ((g \circ_\beta h) \circ_\alpha \widetilde{g}, \overline{(g \circ_\beta h) \circ_\alpha \widetilde{g}} \circ_\beta g \circ_\beta h)$$

where $g \circ_\alpha \widetilde{g} = g \circ_\beta \overline{g} = 1_G$.

These can be written out in terms of ψ_α, ψ_β.

Outline

$g \circ_\beta h = g\psi_\beta(g)h\psi_\beta(g)^{-1}$

Using $\psi \in \mathrm{CC}(G), \beta \in \mathscr{E}$ we get a regular, G-stable subgroup $N \leq \mathrm{Perm}(G)$ in a way analogous to what we had previously: $N = \{\eta_g^{(\beta)} : g \in G\}$ with

$$\eta_g^{(\beta)}[h] = g \circ_\beta h = g\psi_\beta(g)h\psi_\beta(g)^{-1}.$$

That is,

$$\begin{aligned} \eta_g^{(\beta)} &= \lambda(g\psi_\beta(g))\rho(\psi_\beta(g)) \\ &= \lambda(g)C(\psi_\beta(g)), \end{aligned}$$

with $C : G \to \mathrm{Inn}(G)$ being the conjugation map.

Hopf-Galois structure: $\eta_g^{(\beta)}[h] = g\psi_\beta(g)h\psi_\beta(g)^{-1}$

Let L/K be a Galois extension with $\mathrm{Gal}(L/K) = G$.

Let $N = N_\beta$ be as above (depending on ψ, β).

Then G acts on $L[N]$ by

$$^{k}(\ell\eta_g^{(\beta)}) = k(\ell)\eta^{(\beta)}_{kg\psi_\beta(g)k^{-1}\psi_\beta(g)^{-1}},\ g, k \in G,\ \ell \in L.$$

Let $H = L[N]^G$. Then L/K is an H-Galois extension.

So L/K has Hopf-Galois structures of type isomorphic to $(G, \circ_\beta)$ for all $\beta \in \mathscr{E}$.

Fact. $\mathrm{Gp\text{-}Like}(H) = \{\eta_g^{(\beta)} \in N : g\psi_\beta(g) \in Z\} = \{\rho(g^{-1}) : g\psi_\beta(g) \in Z\}$.

More HGS

Since $(G, \circ_\alpha, \circ_\beta)$ is a brace, we have more Hopf-Galois structures, though not necessarily on the same extension L/K.

We have $N_\beta = \{\eta_g^{(\beta)} : g \in G\} \leq \mathrm{Perm}(G) = \mathrm{Perm}(G, \circ_\alpha)$ is regular.

It is also $(G, \circ_\alpha)$-stable: ${}^k\eta_g^{(\beta)} = \eta^{(\beta)}_{k \circ_\alpha (g \circ_\beta \widetilde{k})} = \eta^{(\beta)}_{(k \circ_\alpha h) \circ_\beta \overline{k}}, \; k \circ_\alpha \widetilde{k} = 1_G$.

For fixed $\beta \in \mathscr{E}$, $N_\beta \leq \mathrm{Perm}(G, \circ_\alpha)$ is regular, $(G, \circ_\alpha)$-stable and hence yields a Hopf-Galois structure on each $(G, \circ_\alpha)$-Galois extension, $\alpha \in \mathscr{E}$.

Outline

Nilpotency class two

We say G (still nonabelian) has *nilpotency class two* if $[G, G] \leq Z$.

Examples:

- D_4, dihedral group, order 8: $[G, G] = Z = \langle r^2 \rangle$;
- Q_8, quaternion group: $[G, G] = Z$ is the subgroup of order 2;
- $H(p)$, the Heisenberg group mod p: $[G, G] = Z$ cyclic, order p;
- extraspecial groups: p-groups with $Z \cong C_p$ and $G/Z \cong C_p^{n-1}$ ($|G| = p^n$).

Nilpotency class two

We have $\psi = 1 \in \mathrm{CC}(G)$, and *there is no reason to consider any other choice of* ψ.

The brace block with $\psi = 1$ will contain every brace block on G starting with some choice of $\psi \in \mathrm{CC}(G)$.

On opposites

Recall: for $(B, \cdot, \circ)$ a brace we have $(B, \cdot', \circ)$ is also a brace, where $a \cdot' b = b \cdot a$ for all $a, b \in B$. We call $(B, \cdot', \circ)$ the *opposite* brace to $(B, \cdot, \circ)$.

Example

The almost trivial brace is the opposite brace to the trivial brace (and vice versa).

If we choose $\psi = 1$ then we can pick $\alpha = -1$ and obtain

$$g \circ_\alpha h = g\alpha(g)h\alpha(g)^{-1} = gg^{-1}hg = hg = g \cdot' h.$$

As choosing $\alpha = 0 \in \mathscr{E}$ always gives the trivial brace, we obtain:

If G has nilpotency class two, then any maximal brace block contains both the trivial brace and the almost trivial brace on G.

An equivalent opposite

What about opposites for $(G, \cdot, \circ_\alpha)$?

A general observation

Given $(B, \cdot, \circ)$ define a binary operation $\hat{\circ}$ by

$$a \,\hat{\circ}\, b = (a^{-1} \circ b^{-1})^{-1},\ a, b \in B.$$

Then $(B, \cdot, \hat{\circ})$ is a brace, and $(B, \cdot, \hat{\circ}) \cong (B, \cdot', \circ)$ via $a \mapsto a^{-1}$.

If G is any finite nonabelian group, $\psi \in \mathrm{CC}(G),\ \alpha \in \mathscr{E}$ we have

$$\begin{aligned} g \,\hat{\circ}\, h &= (g^{-1} \circ_\alpha h^{-1})^{-1} \\ &= (g^{-1}\psi_\alpha(g^{-1})h^{-1}\psi_\alpha(g^{-1})^{-1})^{-1} \\ &= \psi_\alpha(g^{-1})h\psi_\alpha(g^{-1})^{-1}g \end{aligned}$$

$g \,\hat{\circ}\, h = \psi_\alpha(g^{-1}) h \psi_\alpha(g^{-1})^{-1} g$

Assume G has nilpotency class two, and let $\psi = 1 \in \mathrm{CC}(G)$.
So

$$g \,\hat{\circ}_\alpha\, h = \alpha(g^{-1}) h \alpha(g^{-1})^{-1} g.$$

For $\alpha = \phi_1 + \cdots + \phi_t \in \mathscr{E}$, let $\alpha^* = \phi_t + \cdots + \phi_1 \in \mathscr{E}$.
Let $\beta = -1 + \alpha(-1) = -1 - \alpha^* \in \mathscr{E}$. Then $\beta(g) = g^{-1}\alpha(g^{-1})$ and

$$\begin{aligned} g \circ_\beta h &= g\beta(g)h\beta(g)^{-1} \\ &= g(g^{-1}\alpha(g^{-1}))h(g^{-1}\alpha(g^{-1}))^{-1} \\ &= \alpha(g^{-1})h\alpha(g^{-1})^{-1}g \\ &= g \,\hat{\circ}_\alpha\, h \end{aligned}$$

If $\psi = 1$ then $(G, \cdot, \circ_\alpha)$ is in a brace block if and only if $(G, \cdot, \hat{\circ}_\alpha)$ is.

This does not happen for general ψ (hence, for general G).

Example: Q_8

Let L/K be a Galois extension, Galois group $G = Q_8$.

Write $Q_8 = \langle a, b : a^4 = b^4 = a^2b^2 = abab^{-1} = 1_G \rangle$.

We will cast the results to follow in terms of regular subgroups rather than braces.

Regular subgroups of $\mathrm{Perm}(Q_8)$ are classified in [Taylor-Truman, 2019]. There are 22 subgroups:

Type $C_2 \times C_2 \times C_2$: 2 structures
Type $C_4 \times C_2$: 6 structures
Type C_8: 6 structures
Type Q_8: 2 structures
Type D_4: 6 structures

Some endomorphisms

Let $s, t \in \{a, b, ab\},\ s \neq t$. Consider the following elements of $\mathrm{End}(G)$:

	ϕ_0	ϕ_1	ϕ_2	ϕ_3	ϕ_4
s	1_G	s^3t	s^2t	t	s^3
t	1_G	s^3	s^3	st	st

For $\alpha \in \mathscr{E}$ given by:

1. $\alpha = \phi_0 = 0 \Rightarrow N = \lambda(G) \cong G$.
2. $\alpha = 1 \Rightarrow N = \rho(G) \cong G$.
3. $\alpha = \phi_1 \Rightarrow N = \langle \lambda(s)\rho(t), \lambda(s^2), \lambda(t)\rho(st) \rangle \cong C_2 \times C_2 \times C_2$.
4. $\alpha = \phi_2 + \phi_3 \Rightarrow N = \langle \lambda(s), \rho(t) \rangle \cong C_4 \times C_2$.
5. $\alpha = \phi_4 \Rightarrow N = \langle \rho(s), \lambda(s)\rho(t) \rangle \cong D_4$.
6. $\alpha = -1 - \phi_4 \Rightarrow N = \langle \lambda(s), \lambda(t)\rho(s) \rangle \cong D_4$.

It turns out that by varying s, t we get all regular, G-stable subgroups of $\mathrm{Perm}(Q_8)$ except those of cyclic type.
This gives rise to a brace block with 16 different operations.

A brace block of size 16

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
s	s^3t	s^2t	t	s^3
t	s^3	s^3	st	st

$N = \{\lambda(g\alpha(g))\rho(\alpha(g)) : g \in G\}$

1. $\alpha = 0 \Rightarrow N = \lambda(G) \cong G$.
2. $\alpha = 1 \Rightarrow N = \rho(G) \cong G$.
3. $\alpha = \phi_1 \Rightarrow N = \langle \lambda(s)\rho(t), \lambda(s^2), \lambda(t)\rho(st) \rangle \cong C_2 \times C_2 \times C_2$.
4. $\alpha = \phi_2 + \phi_3 \Rightarrow N = \langle \lambda(s), \rho(t) \rangle \cong C_4 \times C_2$.
5. $\alpha = \phi_4 \Rightarrow N = \langle \rho(s), \lambda(s)\rho(t) \rangle \cong D_4$.
6. $\alpha = -1 - \phi_4 \Rightarrow N = \langle \lambda(s), \lambda(t)\rho(s) \rangle \cong D_4$.

Remarks.

- This block is maximal: we cannot pick α to get $N \cong C_8$ since $|\lambda(g\psi_\alpha(g))\rho(\psi_\alpha(g))| \leq \mathrm{lcm}(|g\psi_\alpha(g)|, |\psi_\alpha(g)|)$.
- Neither (4) nor (6) above can be obtained with some $\alpha \in \mathrm{End}(Q_8)$.

Outline

The opposite

Recall: if G has nilpotency class two, then $\hat{\circ}_\alpha$ is in the brace block $(G, \{\circ_\alpha : \alpha \in \mathscr{E}\})$ (setting $\psi = 1$).

For general G, there are other cases where this may occur, a simple example being $(G, \{\cdot, \cdot'\})$.

If the nilpotency class is greater than 2, is there a condition to determine when $\hat{\circ}_\alpha$ will be in a brace block containing $(G, \cdot, \circ_\alpha)$?

Such a condition would presumably depend not just on G but on the particular choice of ψ.

Under what conditions will a brace block contain both $(G, \circ_\alpha, \circ_\beta)$ and $(G, \circ'_\alpha, \circ_\beta)$ (or $(G, \circ_\alpha, \widehat{\circ}_\beta)$)?

Do we find all brace blocks with underlying group G?

No.

Example

Let $G = S_n,\ n \geq 5$. We have $[S_n, S_n] = A_n,\ Z = \{\iota\}$.
If $\psi \in \mathrm{CC}(G)$ then $A_n \subseteq \ker\psi$.
So

$$\psi(\sigma) = \begin{cases} \iota & \sigma \in A_n \\ \tau & \sigma \notin A_n \end{cases},\ \tau \in S_n,\ \tau^2 = \iota.$$

Let $\psi, \psi' \in \mathrm{CC}(G)$ be such that $\psi(G) = \langle(12)\rangle,\ \psi'(G) = \langle(34)\rangle$.
Since $\psi_\alpha(G) \subseteq \psi(G) = \langle(12)\rangle$ we see that $\psi' \neq \psi_\alpha$ for any $\alpha \in \mathscr{E}$.
So the circle operations given by ψ, ψ' do not appear in the same brace block.

To reinterate: no

$\psi(G) = \langle(12)\rangle := \langle\tau\rangle,\ \psi'(G) = \langle(34)\rangle := \langle\tau'\rangle$

Denote the corresponding binary operations by $\circ$ and $\star$ respectively.

Clearly, $(G, \cdot, \circ)$ and $(G, \cdot, \star)$ are biskew braces (since $\psi, \psi' \in \mathrm{Ab}(G)$).

However, we can show that $(G, \circ, \star)$ is a biskew brace as well: it follows from the fact that $\tau\tau' = \tau'\tau$.

Thus, $(G, \{\cdot, \circ, \star\})$ is a brace block.

This quickly generalizes to brace blocks with up to $2^{\lfloor n/2 \rfloor}$ groups, each isomorphic to S_n or $A_n \times C_2$.

Can this be extended in a reasonable way to find all brace blocks?

If so, it would also find all bi-skew braces.

Opportunity: HGS of abelian type

In the classic, recursive constructions, only one abelian group arises in a brace block: if $(G, \circ_n)$ is abelian, then $(G, \circ_{n+1}) = (G, \circ_n)$.

Thus we could not obtain any nontrivial braces $(B, \cdot, \circ)$ with both $(B, \cdot)$ and $(B, \circ)$ abelian.

Equivalently, we could not use the theory to find Hopf-Galois structures on an abelian extension of abelian type.

It is possible now.

Example

Return to $G = Q_8 = \langle a, b \rangle$. Let $\phi_1(a) = a^3 b,\ \phi_1(b) = a^3,\ \phi_2(a) = a^2 b,\ \phi_2(b) = a^3,\ \phi_3(a) = b,\ \phi_3(b) = ab$.

Let $\alpha = \phi_1$ and $\beta = \phi_2 + \phi_3$.

Then $(G, \circ_\alpha) \cong C_2 \times C_2 \times C_2$ and $(G, \circ_\beta) \cong C_4 \times C_2$.

This gives a HGS on a $C_2 \times C_2 \times C_2$ extension of type $C_4 \times C_2$ and vice versa.

Problem: $\mathscr{E}$ is messy

$\mathscr{E}$ is very large.

Example

There are 30 HGS on an S_5-extension [Carnahan-Childs, 99], hence at most 30 bi-skew braces $(G, \cdot, \circ)$ with $(G, \cdot) = S_5$.
Let $\psi = 1$.
$\mathrm{End}(S_5)$ has 146 elements, including 120 automorphisms.
So $\mathscr{E}$ contains, among other things, all sums of elements of $\mathrm{End}(S_5)$.
So $\mathscr{E}$ has many more than $2^{146} \approx 10^{44}$ elements.

In general, many $\alpha \in \mathscr{E}$ give identical braces.
It would be good to have an effective way to pick "different" α's.
Say $\mathscr{E}/\sim$, where $\alpha \sim \beta \Rightarrow \psi(\alpha - \beta) \subset Z$.

Thank you.