Hopf-Galois module structure of tame radical extensions of square-free degree

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Some historical results

Theorem (Noether, 1932)

If L/K is a tamely ramified Galois extension of number fields with Galois group G, then $\mathcal{O}_L$ is a locally free $\mathcal{O}_K G$-module (of rank one).

Remark

In general, criteria for global freeness are more delicate.

- Del Corso and Rossi (2013) determined criteria for global freeness for L/K a tame Kummer extension (see the following slide)
- Truman (2020) studied a non-normal analogue of the result of Del Corso and Rossi for tamely ramified extensions of prime degree using Hopf-Galois theory

Aim

Our main aim is to generalise the work of Truman to certain families of tamely ramified extensions of square-free degree which have a unique almost classical Hopf-Galois structure.

The result of Del Corso and Rossi

Let L/K be a tamely ramified Kummer extension of exponent m and degree N.

Definition

For $\alpha_1, ..., \alpha_r$ a set of Kummer generators for L/K, define $a_i = \alpha_i^m$, and write $\boldsymbol{\alpha}$ for $(\alpha_1, ..., \alpha_r)$ and $\boldsymbol{a}$ for $(a_1, ..., a_r)$. Similarly if $i_1, ..., i_r \in \mathbb{N}$ write $\boldsymbol{i}$ for $(i_1, ..., i_r)$.

Theorem (Del Corso and Rossi, 2013)

The extension L/K has a normal integral basis iff there exists a set of integral Kummer generators $\boldsymbol{\alpha}$ such that the following conditions hold.

- *The ideals $\mathcal{B}_{\boldsymbol{i}} = \prod_{\mathfrak{p}} \mathfrak{p}^{\lfloor \frac{v_{\mathfrak{p}}(\boldsymbol{a}^{\boldsymbol{i}})}{m} \rfloor}$ are principal for all $\boldsymbol{i}$.*
- *The congruence $\sum_{\boldsymbol{i}} \frac{\boldsymbol{\alpha}^{\boldsymbol{i}}}{x_{\boldsymbol{i}}} \equiv 0 \pmod{N}$ holds for some $x_{\boldsymbol{i}} \in \mathcal{O}_K$ with $\mathcal{B}_{\boldsymbol{i}} = x_{\boldsymbol{i}}\mathcal{O}_K$.*

Further, when this is the case, the integer $\omega = \frac{1}{N}\sum_{\boldsymbol{i}} \frac{\boldsymbol{\alpha}^{\boldsymbol{i}}}{x_{\boldsymbol{i}}}$ generates $\mathcal{O}_L$ over $\mathcal{O}_K G$.

Hopf-Galois module theory

Now let L/K be a finite extension of number fields and suppose L/K is Hopf-Galois for some Hopf algebra H. Using Hopf-Galois module theory, we have a Hopf-Galois analogue of the normal basis theorem.

Theorem

L is a free H-module (of rank one).

Definition

We can define the *associated order* of $\mathcal{O}_L$ in H as

$$\mathcal{A}_H := \{h \in H | h.x \in \mathcal{O}_L \text{ for all } x \in \mathcal{O}_L\}.$$

Hopf-Galois module theory is concerned with the following properties of $\mathcal{A}_H$.

- The structure of $\mathcal{A}_H$ as a ring
- The structure of $\mathcal{O}_L$ as an $\mathcal{A}_H$-module
  - i.e. whether $\mathcal{O}_L$ is locally or globally free as an $\mathcal{A}_H$-module

Setup for the extension

- Let K be a number field
- Let m be an odd square-free positive integer with factorisation $p_1...p_r$
- Let ζ_m be a primitive m^{th} root of unity
- Assume that p is *unramified* in K for all $p|m$
  - Note that this implies that $\zeta_{p_i} \notin K$ for $1 \leq i \leq r$
- Let $L = K(\alpha_1, ..., \alpha_r)$ where each $a_i := \alpha_i^{p_i} \in K$
- Assume that $a_i \in K \backslash K^{p_i}$ for all i
  - This ensures that the extension has degree m
- Assume that if $\mathfrak{p}|p_i\mathcal{O}_K$, then $\mathfrak{p}$ is unramified in $K(\alpha_j)$ for all $j \neq i$
  - This allows us to apply arithmetic disjointness to determine local integral bases for $\mathfrak{p}|m\mathcal{O}_K$
- Let E be the Galois closure of L/K
- It can be shown that $E = L(\zeta_m)$

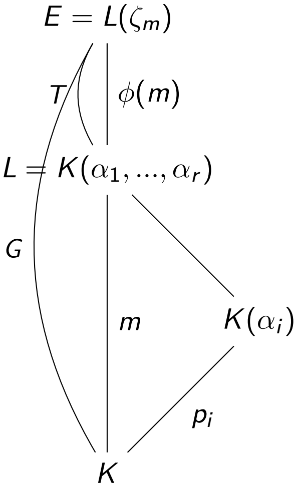
Properties of the Galois group

- Let $G = Gal(E/K)$
- Let $T = Gal(E/L)$
- We have $G = \langle \sigma_1, ..., \sigma_r, \tau_1, ..., \tau_r \rangle$ and $T = \langle \tau_1, ..., \tau_r \rangle$ where

$$\sigma_i(\alpha_i) = \zeta_{p_i}\alpha_i,\ \sigma_i(\alpha_j) = \alpha_j,\ \sigma_i(\zeta_{p_j}) = \zeta_{p_j},$$

$$\tau_i(\alpha_j) = \alpha_j,\ \tau_i(\zeta_{p_i}) = \zeta_{p_i}^{d_i} \text{ and } \tau_i(\zeta_{p_j}) = \zeta_{p_j}$$

  for d_i some primitive root modulo p_i
- $G \cong S \rtimes T$ with $S := \langle \sigma_1, ..., \sigma_r \rangle$



Properties of the Galois group

Lemma

Since $Gal(E/L)$ (the group T) has a normal complement in G (namely S), the extension L/K is almost classically Galois.

Remark

Since it can be shown that S is the unique normal complement to T in G, the extension L/K has a unique almost classical Hopf-Galois structure.

Remark

For $r = 2$, this is the only Hopf-Galois structure admitted by the extension.

Properties of the Hopf-Galois structure

- The subgroup of $Perm(G/T)$ which gives rise to the unique almost classical Hopf-Galois structure is $\lambda(S)$
- The corresponding Hopf algebra is $H = E[\lambda(S)]^G$
- H has a K-basis consisting of mutually orthogonal idempotents

$$e_{\boldsymbol{i}} = \frac{1}{m} \prod_{k=1}^{r} \sum_{n=0}^{p_k - 1} \zeta_{p_k}^{-i_k n_k} \lambda(\sigma_k)^{n_k}$$

- These give rise to an isomorphism of K-algebras, $H \cong K^m$
- H acts on L in the following way

$$e_{\boldsymbol{i}}(\boldsymbol{\alpha}^{\boldsymbol{j}}) = \begin{cases} \boldsymbol{\alpha}^{\boldsymbol{j}} & \text{if } \boldsymbol{i} = \boldsymbol{j} \\ 0 & \text{otherwise} \end{cases}$$

Determining criteria for the extension to be tamely ramified

Lemma

L/K is tame iff all α_i can be chosen to satisfy $a_i := \alpha_i^{p_i} \equiv 1 \pmod{p_i^2 \mathcal{O}_K}$.

Proof.

- Firstly, we apply the standard result that L/K is tame iff the sub-extensions $K(\alpha_i)/K$ are tame for all i.
- Secondly, we apply a result of Truman (2020) that $K(\alpha_i)/K$ is tame iff α_i can be chosen to satisfy $a_i := \alpha_i^{p_i} \equiv 1 \pmod{p_i^2 \mathcal{O}_K}$.

□

Remark

Henceforth, we will assume that these congruences hold.

Determining local integral bases

Definition

We will use $\pi_{\mathfrak{p}}$ to denote a uniformiser in $K_{\mathfrak{p}}$.

Definition

Henceforth we will use q_i to denote the integer $\frac{m}{p_i}$.

For prime ideals $\mathfrak{p} \nmid m\mathcal{O}_K$ we study each sub-extension and use properties of arithmetic disjointness and obtain that a local integral basis is given by

$$\left\{ \frac{\boldsymbol{\alpha}^{\boldsymbol{i}}}{\pi_{\mathfrak{p}}^{\left\lfloor \frac{v_{\mathfrak{p}}\left(\prod_{j=1}^{r} a_j^{i_j q_j}\right)}{m} \right\rfloor}} \right\}$$

Determining local integral bases

For prime ideals $\mathfrak{p} | m\mathcal{O}_K$ we use a different approach

- Truman (2020) determined local integral bases for the prime degree case
- We note that each subextension $K(\alpha_i)/K$ has prime degree
- The assumption that if $\mathfrak{p} | p_i\mathcal{O}_K$, then $\mathfrak{p}$ is unramified in $K(\alpha_j)$ for all $j \neq i$ allows us to apply arithmetic disjointness here
- We obtain that a local integral basis for $\mathfrak{p} | p_i\mathcal{O}_K$ is the "product" of the following sets

$$\left\{1, \alpha_i, \alpha_i^2, ..., \alpha_i^{p_i-2}, \frac{(1 + \alpha_i + ... + \alpha_i^{p_i-1})}{p_i}\right\}$$

$$\left\{\frac{\boldsymbol{\alpha}^{\boldsymbol{j}}}{\pi_{\mathfrak{p}}^{\left\lfloor \frac{v_{\mathfrak{p}}(\prod_{k=1}^{r} a_k^{j_k q_k})}{m} \right\rfloor}} \mid \text{where the } i^{th} \text{ component of } \boldsymbol{j} \text{ is } 0\right\}$$

Determining the associated order

Definition

Let $\mathcal{M}$ denote the unique *maximal order* in H.
For $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$, let $\mathcal{M}_\mathfrak{p}$ denote the unique *maximal order* in $H_\mathfrak{p}$.

Proposition (Truman, 2011)

For prime ideals $\mathfrak{p} \nmid m\mathcal{O}_K$*, we have* $\mathcal{A}_{H,\mathfrak{p}} = \mathcal{M}_\mathfrak{p}$ *and* $\mathcal{O}_{L,\mathfrak{p}}$ *is free over* $\mathcal{A}_{H,\mathfrak{p}}$*.*

In our case $\mathcal{M}_\mathfrak{p} = \mathcal{O}_{K,\mathfrak{p}}\langle(e_{\boldsymbol{i}})\rangle$.

Proposition

For prime ideals $\mathfrak{p}|m\mathcal{O}_K$*,* $\mathcal{O}_{L,\mathfrak{p}}$ *is a free* $\mathcal{A}_{H,\mathfrak{p}}$*-module (of rank one).*

For prime ideals $\mathfrak{p}|m\mathcal{O}_K$ we determine the associated order and prove freeness "all in one". We will sketch the proof of freeness for prime ideals $\mathfrak{p}|m\mathcal{O}_K$ on the following slide.

Determining the associated order

- Let i be such that $\mathfrak{p}|p_i\mathcal{O}_K$
- We fix a particular candidate generator $x_{\boldsymbol{p-1}}$ which is chosen to have the largest power of p_i in the denominator
- We determine elements $a_{\boldsymbol{i}} \in H_{\mathfrak{p}}$ such that $a_{\boldsymbol{i}}.x_{\boldsymbol{p-1}} = x_{\boldsymbol{i}}$ for all integral basis elements $x_{\boldsymbol{i}}$
- Note that these elements exist because our "candidate generator" generates $L_{\mathfrak{p}}$ as an $H_{\mathfrak{p}}$-module
- To determine whether $a_{\boldsymbol{i}} \in \mathcal{A}_{H,\mathfrak{p}}$ we need to evaluate $a_{\boldsymbol{i}}.x_{\boldsymbol{j}}$ for all basis elements $x_{\boldsymbol{j}}$
- $a_{\boldsymbol{i}} \in \mathcal{A}_{H,\mathfrak{p}}$ iff $a_{\boldsymbol{i}}.x_{\boldsymbol{j}} \in \mathcal{O}_{L,\mathfrak{p}}$ for all integral basis elements $x_{\boldsymbol{j}}$
- It turns out that the elements $a_{\boldsymbol{i}}$ are actually in the associated order as claimed
- Hence the elements $a_{\boldsymbol{i}}$ form an $\mathcal{O}_{K,\mathfrak{p}}$-basis of the associated order and $\mathcal{O}_{L,\mathfrak{p}} = \mathcal{A}_{H,\mathfrak{p}}.x_{\boldsymbol{p-1}}$
- In fact a stronger result is true here, we have $a_{\boldsymbol{i}} \in \mathcal{O}_L[\lambda(S)]^G$

Using idèlic theory to derive conditions for global freeness

The main result that we will use to derive conditions for global freeness is the following.

Theorem (Bley and Johnston, 2008)

$\mathcal{O}_L$ is a free $\mathcal{A}_H$-module iff

- *$\mathcal{O}_L$ is a locally free $\mathcal{A}_H$-module*
- *$\mathcal{M}\mathcal{O}_L$ is a free $\mathcal{M}$-module with a generator $x \in \mathcal{O}_L$.*

We have shown that $\mathcal{O}_L$ is a *locally* free $\mathcal{A}_H$-module.
To determine when $\mathcal{M}\mathcal{O}_L$ is free $\mathcal{M}$-module with a generator $x \in \mathcal{O}_L$, we use the idèlic description of $Cl(\mathcal{M})$ (the locally free class group of $\mathcal{M}$).

- $\mathcal{M}\mathcal{O}_L$ is a free $\mathcal{M}$-module iff $\mathcal{M}\mathcal{O}_L$ has trivial class in $Cl(\mathcal{M})$
- The isomorphism $H \cong K^m$ gives rise to an isomorphism of class groups
- $Cl(\mathcal{M}) \cong \frac{\mathbb{J}(H)}{H^\times \mathbb{U}(\mathcal{M})} \cong Cl(\mathcal{O}_K)^m$

Using idèlic theory to derive conditions for global freeness

- The class of $\mathcal{M}\mathcal{O}_L$ corresponds to the tuple $(\mathcal{B}_{\boldsymbol{i}})$ where

$$\mathcal{B}_{\boldsymbol{i}} = \prod_{\mathfrak{p}} \mathfrak{p}^{\lfloor \frac{v_{\mathfrak{p}}(\boldsymbol{a}^{\boldsymbol{i}})}{m} \rfloor}$$

- $\mathcal{M}\mathcal{O}_L$ is a free $\mathcal{M}$-module with a generator $x \in \mathcal{O}_L$ iff the ideals $\mathcal{B}_{\boldsymbol{i}}$ are principal with generators $x_{\boldsymbol{i}}$ such that $\sum_{\boldsymbol{i}} \frac{\boldsymbol{\alpha}^{\boldsymbol{i}}}{x_{\boldsymbol{i}}} \equiv 0 \pmod{m\mathcal{O}_L}$

Our conclusion is the following.

Theorem

$\mathcal{O}_L$ is a free $\mathcal{A}_H$-module iff there exist $\alpha_1, ..., \alpha_r \in \mathcal{O}_L$ such that

- $L = K(\alpha_1, ..., \alpha_r)$
- $a_i := \alpha_i^{p_i} \in \mathcal{O}_K$ *for all* $1 \leq i \leq r$
- *The ideals $\mathcal{B}_{\boldsymbol{i}}$ as defined above are principal with generators $x_{\boldsymbol{i}}$ such that* $\sum_{\boldsymbol{i}} \frac{\boldsymbol{\alpha}^{\boldsymbol{i}}}{x_{\boldsymbol{i}}} \equiv 0 \pmod{m\mathcal{O}_L}$

Furthermore, in this case the element $\frac{1}{m}\sum_{\boldsymbol{i}} \frac{\boldsymbol{\alpha}^{\boldsymbol{i}}}{x_{\boldsymbol{i}}}$ is a free generator of $\mathcal{O}_L$ as an $\mathcal{A}_H$-module

Rewriting the extension using a single radical generator

Remark

It is possible to describe the extension as $L = K(\delta)$ *with the minimum polynomial of* δ *over* K *being* $x^m - d$ *(so that the extension has degree* m*)*
Using this point of view the extension looks more like a cyclic Kummer extension
Translating to this description gives a cleaner presentation of the final result from which it is easier to see the connection with the result of Del Corso and Rossi
Note that using the description of $L = K(\alpha_1, ..., \alpha_r)$ *eases obtaining the conditions for tameness and the calculations used to determine the local integral bases*

Further work

- Multiple m^{th} roots (with m square-free)
  - i.e. study extensions of the form $L = K(\alpha_1, ..., \alpha_r)$ where each $\alpha_i^m \in K$
- Single p^r root
  - i.e. study extensions of the form $L = K(\alpha)$ where $\alpha^{p^r} \in K$
- Work towards a complete analogue of the Del Corso and Rossi result

Thank you for your attention.