HOPF ALGEBRAS AND GALOIS MODULE THEORY, MAY 30 – JUNE 3, 2022

Half-baked ideas are welcome! This has always been a working conference, and over the years, one of the most enjoyable features has been the inclusion of half-baked ideas, guiding principles, and crazy conjectures. Please come with some to share.

Breakout rooms provided. Another valuable feature of this conference has always been the ample time provided for discussion. This year, there will be breakout rooms provided after every two talks. One room named for the first talk, one for the second, and one for general conversation. I want to thank Ilaria Colazzo the suggestion.

Zoom link: `unomaha.zoom.us/j/98718137677?pwd=TTNLREZibHNsZStGalZMTlFnZTdFdz09`
We intend to record the talks and post them at `www.hopf-galois.org/`. If you would rather not have your talk recorded, please let me know.

Time Zone Conversion. You might find this helpful. I did.

CDT	EDT	**UTC**	BST	CEST	JST
8:00	9:00	**13:00**	14:00	15:00	22:00
12:00	13:00	**17:00**	18:00	19:00	2:00

Monday. Moderator: Andrea Caranti

13:00UTC Cindy Tsang, *Characterization of the type of Hopf-Galois structures on cyclic extensions.* 25 minutes
13:30UTC Tim Kohl, *Normalizing Graphs of Regular Permutation Groups.* 50 minutes
BREAK 14:30UTC – Breakout rooms available.
15:00UTC Ilaria Del Corso, *On Fuchs' problem on the group of units of a ring: the state of the art and some progress using braces.* 50 minutes
16:00UTC Rob Underwood, *Galois Extensions, Forms, and Hopf-Galois Theory.* 50 minutes
BREAK 17:00UTC – Breakout rooms available.

Tuesday. Moderator: Tim Kohl

13:00UTC Lorenzo Stefanello, *Some new examples of Hopf–Galois structures in which the Hopf–Galois correspondence is surjective.* 50 minutes
14:00UTC Senne Trappeniers, *Bi-skew braces and brace blocks.* 50 minutes
BREAK 15:00UTC – Breakout rooms available.
15:30UTC Andrea Caranti, *Skew braces that do not come from Rota–Baxter operators.* 25 minutes
16:00UTC Alan Koch, *Commutator-central maps, brace blocks, and Hopf-Galois extensions.* 50 minutes
BREAK 17:00UTC – Breakout rooms available.

Wednesday. Moderator: Nigel Byott

13:00UTC Kevin Keating, *A converse to the Hasse-Arf theorem.* 50 minutes
14:00UTC Paul Schwartz, *Galois scaffolds and Galois module structure for totally ramified extra-special p-extensions.* 25 minutes
BREAK 14:30UTC – Breakout rooms available.
15:00UTC Daniel Gil-Muñoz, *Hopf-Galois module structure of degree p extensions of p-adic fields.* 50 minutes
16:00UTC Lindsay N. Childs *A skew left brace yields a solution of the Yang-Baxter equation.* 25 minutes
BREAK 16:30UTC – Breakout rooms available.

Thursday. Moderator: Paul Truman

13:00UTC Andrea Caranti and Cindy Tsang, *Finite p-groups of class two with a very large multiple holomorph.* 25 minutes

13:30UTC Nigel Byott, *Insoluble subgroups of the holomorph of a finite soluble group.* 50 minutes

BREAK 14:30UTC – Breakout rooms available.

15:00UTC Andrew Darlington, *Hopf-Galois Structures on separable field extensions of degree pq.* 50 minutes

16:00UTC Elena Campedel, *Hopf Galois structures on Galois extensions of order p^2q and skew braces of size p^2q.* 25 minutes

BREAK 16:30UTC – Breakout rooms available.

6:00PM CDT Dinner at Cascio's Italian Steakhouse – 1620 S10th St. Catch the ORBT on Dodge at 4:00pm. Walk 1.1 mile (24 min) down S10th Street to restaurant. Visit Durham Western Heritage Museum enroute.

Friday. Moderator: Kevin Keating

Late start

14:00UTC Isabel Martin-Lyons and Paul Truman, *Introducing weak skew braces.* 50 minutes

BREAK 15:00UTC – Breakout rooms available.

15:30UTC Ilaria Colazzo, *YB-semitrusses and left non-degenerate solutions to the Yang-Baxter equation.* 50 minutes

16:30UTC George Prestidge, *Hopf-Galois module structure of tame radical extensions of squarefree degree.* 25 minutes

BREAK 17:00UTC – Breakout rooms available.

Abstracts

Nigel Byott, University of Exeter.

Insoluble subgroups of the holomorph of a finite soluble group. 50 minutes

Abstract: It is an open question whether the holomorph $\mathrm{Hol}(N)$ of a finite soluble group N can contain an insoluble regular subgroup G. A negative answer to this question would mean that (i) any Hopf-Galois structure on an insoluble Galois extension must have insoluble type, and (ii) any finite skew brace with soluble additive group also has soluble multiplicative group. If, instead of requiring G to be regular, we ask that G is an insoluble transitive subgroup of $\mathrm{Hol}(N)$ with soluble point stabilisers, then such pairs (G, N) do exist, but are very special. Indeed, there is a unique minimal example (in the sense that we cannot get a smaller example by passing to subgroups of G and N), namely $(G, N) = (\mathrm{GL}_3(2), C_2 \times C_2 \times C_2)$, where G is the simple group of order 168. I will outline the proof of this result and discuss its implications for Hopf-Galois structures.

Elena Campedel, Università degli Studi di Milano-Bicocca.

Hopf Galois structures on Galois extensions of order p^2q and skew braces of size p^2q. 25 minutes

Abstract: In this talk I will present the approach we developed to enumerate all the Hopf Galois structures on Galois extensions of order p^2q and skew braces of size p^2q. This approach is based on the so-called gamma functions, namely maps $\gamma : G \to \mathrm{Aut}(G)$, where G is a group, satisfying $\gamma(x^{\gamma(y)} \cdot y) = \gamma(x)\gamma(y)$. These maps are related both to the regular subgroups of the holomorph of G (and hence to Hopf Galois structures), and to the structures of skew brace on G. Part of the talk will be devoted to explain these connections, and then I present some of the basic results we obtained, which are useful in counting Hopf Galois structures and skew braces.

Joint with Andrea Caranti, Ilaria Del Corso

Andrea Caranti, University of Trento.
Skew braces that do not come from Rota–Baxter operators. 25 minutes
Abstract: Rota–Baxter operators for various kinds of algebras have been studied by several authors since G. Baxter introduced them for commutative algebras in 1960.

Recently, L. Guo, H. Lang, and Y. Sheng introduced Rota–Baxter operators for groups. These were studied further by V. G. Bardakov and V. Gubarev.

It is immediate to see that a Rota–Baxter operator determines a skew brace, via what is called a lambda function in the literature on skew braces. The lambda functions on a group G that come from Rota–Baxter operators are functions from G to its group of inner automorphisms.

We show that to a lambda function on a group G that takes values in the group of inner automorphisms of G one can associate a certain cohomology class, such that the lambda function comes from a Rota–Baxter operator if and only if this class is trivial.

We show how this approach leads to the construction of examples of lambda functions that take values in the group of inner automorphisms, yet do not come from a Rota–Baxter operator. The cohomological approach allows one to analyse these examples in terms of the splitting of certain group extensions.
Joint with Lorenzo Stefanello

Andrea Caranti, University of Trento & Cindy Tsang, Ochanomizu University.
Finite p-groups of class two with a very large multiple holomorph. 25 minutes
Abstract: The study of the quotient $T(G)$ of the multiple holomorph of a group G by its holomorph has been revived in recent years by Timothy Kohl and other authors. There is some evidence to support the conjecture that if the finite group G has trivial center, then $T(G)$ is an elementary abelian 2-group. When G is a finite p-group, the prime divisors of $|T(G)|$ known so far all came from $p(p-1)$. We are able to show that if H is an arbitrary finite group, then there is a finite p-group of nilpotence class two such that $T(G)$ contains a subgroup isomorphic to H. This is work in progress.

Lindsay N. Childs, University of Albany.
A skew left brace yields a solution of the Yang-Baxter equation. 25 minutes
Abstract: Finding solutions of the Yang-Baxter equation was the primary motivation for the definition of skew left braces. But proofs in the literature that show how to get a YBE solution from a skew left brace or any of its special cases (e. g. a brace or a radical algebra) are not particularly clear and often not self-contained. This expository talk will describe a direct, self-contained proof that a skew left brace yields a solution of the YBE. The paper could be viewed as an appendix to Chapter 2 of Childs, Greither, Keating, Koch, Kohl, Truman and Underwood, "Hopf Algebras and Galois Module Theory" (AMS Monographs, vol. 260 (2021).

Ilaria Colazzo, University of Exeter.
YB-semitrusses and left non-degenerate solutions to the Yang-Baxter equation. 50 minutes (could make 25 min work)
Abstract: The study of large classes of set-theoretic solutions can be reduced to the study of some associative algebraic objects, such as braces, skew braces, and YB-semitrusses. A YB-semitruss is an algebraic associative structure that forms a sub-category of semitrusses, allowing one to determine and analyse left non-degenerate solutions to the Yang-Baxter equation. Based on joint work with E. Jespers, A. Van Antwerpen and C. Verwimp, this talk will describe the algebraic structure of YB-semitrusses and its relation with solutions to the Yang-Baxter equation. Then, we will show that skew braces are examples of YB-semitrusses. Finally, we will use YB-semitrusses as a tool to prove that for a finite left non-degenerate solution being right non-degenerate is equivalent to being bijective.

Ilaria Del Corso, Università di Pisa.
On Fuchs' problem on the group of units of a ring: the state of the art and some progress using braces. 50 minutes.
Abstract: L. Fuchs in [Abelian groups, 3rd edn (Pergamon, Oxford, 1960); Problem 72] posed the following problem:
Characterize the groups which are the groups of all units in a commutative and associative ring with identity.

In the following years, these questions inspired the work of many authors, and some partial answer to them has been given. Among the others, we recall the work by Gilmer (1963), Hallett, Hirsch and Zassenhauss (1965-66), Pearson and Schneider (1970) , Dolžan (2002) and the recent papers by Chebolu and Lockridge (2015-17).

In two joint papers with R. Dvornicich (AMPA 2018, BLMS 2018), we studied the original question of Fuchs obtaining a pretty good, but not exhaustive, description of the possible groups of units equipped with families of examples of both realizable and non-realizable groups.

In this talk I will present these results together with some very recent progress I obtained by developing some new tool for braces, which generalizes Thm 1 of the paper by Featherstonhaugh, Caranti and Childs (TAMS 2012). I will give some details also on this part that might be of independent interest.

If I have time I will mention the theorem of classification of *finitely generated* abelian group of units which arise in the interesting case of torsion-free rings (idc JLMS 2020).

Andrew Darlington, University of Exeter.
Hopf-Galois Structures on separable field extensions of degree pq. 50 minutes (could make 25 min work)
Abstract: This talk follows on from the talk Nigel Byott gave last year by beginning to answer the question of what generalising the work on Hopf-Galois structures on non-normal extensions of degree related to a Sophie Germain prime looks like. For example, we discuss how the results generalise to the case of a separable (but not necessarily normal) field extension of degree pq, now for general distinct odd primes p and q, as well as asking the analogous question of how the other degree pq subextensions behave in terms of admitting Hopf-Galois structures. The talk will also outline potential stumbling blocks and possible necessary sacrifices when generalising further in this way, including giving initial examples in this direction.

Daniel Gil-Muñoz, Charles University in Prague.
Hopf-Galois module structure of degree p extensions of p-adic fields. 50 minutes
Abstract: Let p be an odd prime number. The problem of determining the Galois module structure of the ring of integers $\mathcal{O}_L$ in a cyclic degree p extension of p-adic fields L/K was completely solved by F. Bertrandias, J.P. Bertrandias and M.J. Ferton. Their findings characterize the freeness of $\mathcal{O}_L$ over its associated order in terms of the ramification number t of L/K, and more accurately, the continued fraction expansion of $\frac{t}{p}$. We shall present the main ideas behind their proof and show how these techniques can be adapted to describe the Hopf-Galois module structure of the ring of integers $\mathcal{O}_L$ in a degree p extension of p-adic fields L/K whose normal closure $\widetilde{L}$ is the dihedral group D_p of $2p$ elements.

Kevin Keating, University of Florida.
A converse to the Hasse-Arf theorem. 50 minutes
Abstract: Let L/K be a finite Galois extension of local fields with Galois group G. The Hasse-Arf theorem states that if G is abelian then the upper ramification breaks of L/K are all integers. We prove the following partial converse to the Hasse-Arf theorem: Let $p > 2$ and let G be a finite group which is the Galois group of some totally ramified extension of local fields with residue characteristic p. Then there exists a totally ramified extension of local fields L/K with residue characteristic p which has at least one nonintegral upper ramification break.
Joint with Griff Elder

Alan Koch, Agnes Scott College.
Commutator-central maps, brace blocks, and Hopf-Galois extensions. 50 minutes
Abstract: Let L/K be a nonabelian Galois extension, and let $G = \mathrm{Gal}(L/K)$. Let $\psi : G \to G$ be an endomorphism such that $\psi(G)$ is abelian. At Omaha 2021 we showed how ψ can be used to generate a (bi-skew) brace block, which in turn provided for Hopf-Galois structures on L/K (as well as on other Galois extensions with the same degree but with possibly different Galois groups), and non-degenerate, set-theoretic solutions to the Yang-Baxter equation.

In this talk, we will expand upon these results. Let $C, Z \leq G$ denote the commutator subgroup and center of G respectively. Following Caranti-Stefanello we generalize from abelian maps to endomorphisms $\psi : G \to G$ with $\psi(C) \leq Z$. We show how to use ψ to construct a brace block which is considerably larger than the blocks previously considered. As an application, we show how our construction can be used to find all bi-skew braces $(B, \cdot, \circ)$ with at least one of the underlying groups isomorphic to the quaternion group Q_8; this also gives all Hopf-Galois structures in the case $G = Q_8$ except those of cyclic type.

Tim Kohl, Boston University.
Normalizing Graphs of Regular Permutation Groups. 50 minutes
Abstract: For a given group G, embedded in $B = Perm(G)$ as the left regular representation, the normalizer of $\lambda(G)$ in B is the holomorph $\mathrm{Hol}(G)$, and its normalizer $\mathrm{NHol}(G) = \mathrm{Norm}_{\mathrm{B}}(\mathrm{Hol}(G))$ which is called the multiple holomorph. And associated to the multiple holomorph is the collection $\mathcal{H}(G)$ of regular subgroups of $\mathrm{Hol}(G)$ with the property that the normalizer of each $N \in \mathcal{H}(G)$ is $\mathrm{Hol}(G)$ too. Moreover, as is known, $\mathcal{H}(G)$ is the orbit of $\lambda(G)$ under the action of $\mathrm{NHol}(G)$ with stabilizer $\mathrm{Hol}(G)$. As the elements of $\mathcal{H}(G)$ all have the same normalizer, they all mutually normalize each other. We look at a (frequently larger) class of subgroups of $\mathrm{Hol}(G)$ which also have this mutually normalizing property. This collection, denoted $\mathcal{Q}(G)$, is also the orbit of $\lambda(G)$ under the action of a group containing $\mathrm{Hol}(G)$ as the stabilizer. We call this group the quasiholomorph and it is an extension of $\mathrm{Hol}(G)$ which generalizes the notion of the multiple-holomorph, in particular since it contains $\mathrm{NHol}(G)$, but when larger is quite different from $\mathrm{NHol}(G)$ in that it is not a group extension of $\mathrm{Hol}(G)$, but rather is frequently a Zappa-Szép product with the holomorph. The collection $\mathcal{Q}(G)$ contains $\mathcal{H}(G)$ and satisfies the condition of its members being mutually normalizing, and we consider also the graph theoretic implications, namely that one may construct a graph whose vertices are regular subgroups of $\mathrm{Hol}(G)$ where an edge is present if a given subgroup normalizes another. In this setting, the members of $\mathcal{Q}(G)$ form a complete subgraph or 'clique' within this 'normalizing graph'.

Isabel Martin-Lyons and Paul Truman, Keele University.
Introducing weak skew braces. 50 minutes
Abstract: The correspondence between Hopf-Galois structures on Galois field extensions and skew braces revolves around the Greither-Pareigis classification theorem and Byott's translation theorem. However, these results also apply to separable, but non-normal, field extensions; we generalize the correspondence to this setting by defining a new algebraic object, which we call a *weak skew brace*. We develop elementary properties of this object, construct some families of examples, and explore implications for Hopf-Galois theory.

George Prestidge, Keele University.
Hopf-Galois module structure of tame radical extensions of squarefree degree. 25 minutes
Abstract: Noether's theorem tells us that if L/K is a tame, Galois extension of number fields, with Galois group G, then its ring of integers $\mathcal{O}_L$ is locally free over the group ring $\mathcal{O}_K G$. In general, criteria for global freeness are more delicate. Del Corso and Rossi gave such criteria for L/K a tame Kummer extension. We study a non-normal analogue of this situation using Hopf-Galois theory. Truman studied a family of non-normal, tame, radical extensions of prime degree. We generalise the work of Truman to certain families of tame extensions of square-free degree which have a unique almost classical Hopf-Galois structure. We find that $\mathcal{O}_L$ is locally free over its associated order in this Hopf-Galois structure and determine criteria for global freeness. These criteria are identical to those obtained by Del Corso and Rossi in the Galois case.

Paul Schwartz, University of Florida.
Galois scaffolds and Galois module structure for totally ramified extra-special p-extensions. 25 minutes
Abstract: Let K be a local field with perfect residue field. We use MacKenzie-Whaples extensions to construct totally ramified extra-special p-extensions L/K which possess Galois scaffolds. When L/K is such an extension we use the theory of Byott, Childs, and Elder to give sufficient conditions for $\mathfrak{O}_L$ to be free over its associated order $\mathfrak{A}_{L/K} = \{\alpha \in K[\mathrm{Gal}(L/K)] : \alpha\mathfrak{O}_L \subseteq \mathfrak{O}_L\}$. As a special case of this, we construct an extension L/K which satisfies the hypotheses of a theorem by Bondarko and Dievsky and conclude that $\mathfrak{A}_{L/K}$ is a Hopf-Algebra.
Joint with Kevin Keating

Lorenzo Stefanello, Università di Pisa.
Some new examples of Hopf–Galois structures in which the Hopf–Galois correspondence is surjective. 50 minutes
Abstract: Let L/K be a finite Galois extension. Assume that L/K is also H-Galois for a suitable K-Hopf algebra H. It is well known that the correspondence which associates with every sub-Hopf algebra of H an intermediate field between K and L is injective but not necessarily surjective, and apart from some trivial structures, only one class of examples is known where the correspondence is surjective.

It is already of interest to find the failure of the surjectivity, namely the ratio of the number of fields that are reached by the correspondence to the number of all intermediate fields. Some remarkable work in this direction has been carried out in recent years by L. N. Childs, who employed the well-known connection between skew braces and Hopf–Galois structure to translate the problem into the setting of skew braces.

In this talk, after a quick review of the known results, we build on an idea of A. Koch and P. J. Truman to introduce a slightly different connection between skew braces and Hopf–Galois structures. Thanks to this point of view, we present new examples of skew braces which are connected to Hopf–Galois structures where the Hopf–Galois correspondence in surjective.
Joint with Senne Trappeniers.

Senne Trappeniers, Vrije Universiteit Brussel.
Bi-skew braces and brace blocks. 50 minutes
Abstract: Bi-skew braces were defined by L. N. Childs as those skew brace where we can swap the role of the two operations and once again obtain a skew brace. The notion of a brace block was later introduced by A. Koch and is a set with a family of group structures such that any choice of two group structures yields a bi-skew brace.

In this talk we first look at a new characterisation of bi-skew brace, making use of the opposite skew brace. As a consequence of this characterisation we then obtain structural properties of bi-skew braces and in particular, give an affirmative answer to Byott's conjecture for bi-skew braces. Further, we discuss a new explicit construction of brace blocks and see how this relates to the iterative construction given by A. Koch and the recent construction of A. Caranti and L. Stefanello.
Joint with Lorenzo Stefanello.

Cindy Tsang, Ochanomizu University.
Characterization of the type of Hopf-Galois structures on cyclic extensions. 25 minutes
Abstract: We give a complete characterization of the isomorphism classes that can occur as the type of a Hopf-Galois structure on a cyclic extension.

Rob Underwood, Auburn University at Montgomery.
Galois Extensions, Forms, and Hopf-Galois Theory. 50 minutes
Abstract: Let R be a commutative ring with unity and let N be a finitely generated group with finite automorphism group $F = \mathrm{Aut}(N)$. R. Haggenmüller and B. Pareigis have shown that there is a bijection

$$\Theta : \mathcal{G}al(R, F) \to (R[N])$$

from the collection of F-Galois extensions of R to the collection of forms of the Hopf algebra $R[N]$. Let K be a finite field extension of $\mathbb{Q}$ and let E/K be a Galois extension of fields. Let H be the Hopf algebra of

a Hopf-Galois structure on E/K of type N. Then H is an E-form of $K[N]$, and in this case, we show how to construct the preimage of H under Θ. We use the map Θ to determine the Hopf algebra isomorphism classes of the Hopf algebras attached to the Hopf-Galois structures on E/K.
Joint with Timothy Kohl.