Lie trusses (with a bracoid aside)

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Motivation

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- The definition of these brackets involve both an affine and vector spaces.
- Is there a purely affine intrinsic theory of Lie affgebras?

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  - (c) one can shift any pair of points by a rescaled difference between them, i.e., for all $a, b \in A$ and $\lambda \in \mathbb{F}$,
$$a + \lambda \overrightarrow{ab} \in A.$$
- An affine transformation $(A, \vec{A})$ to $(B, \vec{B})$ is a function $f : A \to B$ which induces a linear transformation $\hat{f} : \vec{A} \to \vec{B}$ such that
$$\hat{f}\left(\overrightarrow{ab}\right) = \overrightarrow{f(a)f(b)}.$$

Lie brackets on affine spaces (clasically) [GGU]

A Lie bracket on $(A, \vec{A})$ is an anti-symmetric bi-affine map

$$[-,-]_v : A \times A \longrightarrow \vec{A},$$

that satisfies the Jacobi identity in $\vec{A}$:

$$\widehat{[a,}[b,c]_v]_v + \widehat{[b,}[c,a]_v]_v + \widehat{[c,}[a,b]_v]_v = 0, \qquad (1)$$

where $\widehat{[a,}-]_v$ is the linearisation of the map $[a,-]_v$ etc

Key observation

In the definition of an affine space $(A, \vec{A})$, the vector space $\vec{A}$ is a secondary ingredient and can be got rid of completely.

Heaps [Prüfer '24, Baer '29]

A **heap** is a nonempty set A together with a ternary operation

$$\langle -, -, - \rangle : A \times A \times A \to A,$$

such that for all $a_i \in A$, $i = 1, \ldots, 5$,

(a) $\big\langle \langle a_1, a_2, a_3 \rangle, a_4, a_5 \big\rangle = \big\langle a_1, a_2, \langle a_3, a_4, a_5 \rangle \big\rangle$,

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Homomorphism of heaps: a function $f : A \to B$ such that

$$f\left(\langle a_1, a_2, a_3 \rangle\right) = \big\langle f(a_1), f(a_2), f(a_3) \big\rangle.$$

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- If $(A, +)$ is an (abelian) group, then A is an (abelian) heap with operation
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- $$\mathrm{Aut}(A)\cong \mathrm{Hol}(A_o) = A\rtimes \mathrm{Aut}(A_o),$$
$$f\longmapsto (f(o), f-f(o)).$$

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(G, A_o) **is a bracoid.**

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- G **acts (transitively) on** N **by heap automorphisms.**

Affine spaces (cd) [Breaz, TB, Rybołowicz, Saracco]

An affine space A is a heap with an $\mathbb{F}$-action $(\lambda, a, b) \mapsto \lambda \triangleright_a b$, such that

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Explicitly:

- $\langle a, b, c \rangle = a + \overrightarrow{bc}$;
- $\lambda \triangleright_a b := a + \lambda \overrightarrow{ab}$.

Affine transformations revisited

Where is the vector space?

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The corresponding linear transformation $\hat{f} : A_o \longrightarrow B_o$,

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Affine modules or heaps of modules

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A **heap of modules** over a ring R is a heap A with an R-action $(\lambda, a, b) \mapsto \lambda \triangleright_a b$, such that

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Any abelian heap A is a $\mathbb{Z}$-module in a unique way:

$$n \triangleright_a b = \underbrace{\langle b, a, b, a, \ldots, a, b \rangle}_{2n-1}.$$

Lie affgebras & Lie trusses

Definition

A **(left) Lie bracket** on an affine space A is a bi-affine map $[-,-] : A \times A \to A$ such that, for all $a, b, c \in A$,

$$\langle [a,b], [a,a], [b,a] \rangle = [b,b], \tag{2a}$$

$$\Big\langle [a,[b,c]], [a,a], [b,[c,a]], [b,b], [c,[a,b]] \Big\rangle = [c,c] \tag{2b}$$

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Given an affine space (module) A and a scalar ζ,

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Example

A (non-empty, abelian) heap A is a Lie truss with the brackets, e.g.

$$[a,b] = \langle a,b,a \rangle \quad \text{or} \quad [a,b] = \langle b,a,b \rangle.$$

Relation to the GGU Lie affgebras

Theorem

Let A be an affine space over the field $\mathbb{F}$ ($\mathrm{char}(\mathbb{F}) \neq 2$). For any $o \in A$, there is a bijective correspondence between idempotent Lie brackets on A and vector-valued Lie brackets on (A, A_o).

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$$[a, b]_v = \langle [a, b], b, o \rangle,$$

while in the other

$$[a, b] = [a, b]_v + b.$$

Associative affgebras and trusses

Definition

An affine space with a bi-affine multiplication is called an **affgebra**.

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An affine $\mathbb{Z}$-module with a bi-affine multiplication is simply a **truss**: i.e. an abelian heap with an associative multiplication that distributes over the heap operation.

Lie affgebras of commutators

Theorem

An associative affgebra A is a Lie affgebra with the bracket

$$[a, b] = \langle ab, ba, b\rangle, \tag{3}$$

for all $a, b \in A$.

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An associative affgebra A is a Lie affgebra with the bracket

$$[a, b] = \langle ab, ba, b\rangle, \qquad (3)$$

for all $a, b \in A$.

For all $a \in A$, $D_a = [a, -]$ is a derivation on A along the identity.

Lie affgebras from pre-Lie affgebras

Definition

A *left pre-Lie affgebra* is an affine space A together with the bi-affine map $\cdot : A \times A \longrightarrow A$, such that, for all $a, b, c \in A$,

$$(a \cdot b) \cdot c = \big\langle a \cdot (b \cdot c), b \cdot (a \cdot c), (b \cdot a) \cdot c \big\rangle. \tag{4}$$

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When written in terms of addition $a + b = \langle a, o, b \rangle$, (4) coincide exactly with the pre-Lie algebra conditions.

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When written in terms of addition $a + b = \langle a, o, b \rangle$, (4) coincide exactly with the pre-Lie algebra conditions.

Theorem

Let $(A, \cdot)$ be a right (or left) pre-Lie affgebra. Then A is a Lie affgebra with the bracket

$$[a, b] = \langle a \cdot b, b \cdot a, b \rangle,$$

for all $a, b \in A$.

Derivations on affgebras/trusses

Definition

Let A be an affgebra (truss) and let $\sigma : A \to A$ be an affine (heap) map s.t.

$$\sigma(ab) = \langle \sigma(a)b, \sigma(ab), a\sigma(b) \rangle, \qquad \text{for all } a, b \in A. \tag{5}$$

A **derivation along** σ is an affine (heap) map $X : A \to A$, s.t.,

$$X\sigma = \sigma X, \tag{6a}$$

$$X(ab) = \langle X(a)b, \sigma(ab), aX(b) \rangle, \qquad \text{for all } a, b \in A. \tag{6b}$$

The set of all derivations along σ on A is denoted by $\mathrm{Der}_\sigma(A)$.

Lie bracket as a derivation

Theorem

Let L be a Lie affgebra (truss) with an idempotent bracket, that is, such that, for all $a \in L$,

$$[a, a] = a.$$

Then, for all $a \in L$, $X_a : L \to L$, $b \mapsto [a, b]$, is a derivation on L along the identity.

Lie affgebras of derivations

Theorem

For an affgebra (truss) A, $\mathrm{Der}_\sigma(A)$ is a Lie affgebra (truss) with the affine structure arising from $\mathrm{Aff}(A)$ and the Lie bracket

$$[X, Y] = \langle XY, YX, \sigma \rangle. \tag{7}$$

Lie bracket as a derivation

Theorem

Let L be a Lie affgebra with an idempotent bracket. Then, for all $a \in L$,

$$X_a : L \longrightarrow L, \qquad b \longmapsto [a, b],$$

is a derivation of L along the identity.