Skew bracoids and solutions to the Yang–Baxter

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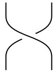
SKEW BRACES, SKEW BRACOIDS,

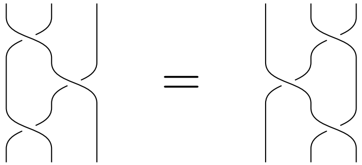
AND HOPF–GALOIS THEORY

Solutions of the Yang-Baxter equation

A (set-theoretic) solution to the YBE is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r). \qquad (*)$$

Write $r =$  . Then $(*)$ becomes



Set-theoretic solutions to the Yang-Baxter equation

Let (X, r) be a solution to the YBE. Write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

- (X, r) is bijective if r is bijective.
- (X, r) is involutive if $r^2 = \mathrm{id}$.
- (X, r) is finite if X is finite.
- (X, r) is right non-degenerate (resp. left non-degenerate) if ρ_x (resp. λ_x) is bijective for all $x \in X$.
- (X, r) is non-degenerate if (X, r) is both left and right non-degenerate.

Examples

X a set.

- $r(x, y) = (x, y)$ is bijective degenerate solution.
- $r(x, y) = (y, x)$ is an bijective non-degenerate solution.
- λ, ρ maps of X. Then $r(x, y) = (\lambda(y), \rho(x))$ is a solution if and only if $\lambda\rho = \rho\lambda$.

  Moreover, (X, r) is right non-degenerate if and only if g is a permutation of X.

  Finally, (X, r) is bijective if and only if (X, r) is non-degenerate.

Examples

G a group

- $r(x, y) = (xy, 1)$ is a left non-degenerate solution.
- $r(x, y) = (y, y^{-1}xy)$ is a bijective non-degenerate solution.
- $r(x, y) = (x^2y, y^{-1}x^{-1}y)$ is a bijective non-degenerate solution.

A crucial example: Radical rings
[Rump, 2007]

Let $(R, +, \cdot)$ be a ring and put $x \circ y = x + xy + y$ for any $x, y \in R$. Then is radical if $(R, \circ)$ is a group.

It $(R, +, \cdot)$ is a radical ring then

$$r(x, y) = (-x + x \circ y, (-x + x \circ y)' \circ x \circ y)$$

is an involutive non-degenerate solution.

Skew braces

A skew brace is a triple $(B, +, \circ)$ such that $(B, +)$ and $(B, \circ)$ are (not necessarily abelian) groups and the following holds

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

for all $a, b, c \in B$.

- $(B, +)$ is the additive structure of $(B, +, \circ)$.
- $(B, \circ)$ is the multiplicative structure of $(B, +, \circ)$.
- The map $\lambda : B \mapsto \mathrm{Aut}(B, +)$ such that $\lambda_a(b) = -a + a \circ b$ is called the λ-map.

Examples.

- Let $(G, +)$ be (any) group. Then $(G, +, +)$ and $(G, +^{op}, +)$ are skew braces.
- Any radical ring is a skew brace.

Skew braces and solutions

Theorem [Guarnieri and Vendramin, 2017]

Let B be a skew brace. Define $r : B \times B \to B \times B$ by

$$r(x, y) = r(-x + x \circ y, (-x + x \circ y)' \circ x \circ y),$$

where a' denotes the inverse of a in $(B, +)$ Then r is a bijective non-degenerate solution of the YBE. Moreover,

$$r \text{ is involutive } \iff (B, +) \text{ is abelian.}$$

The structure group

Let (X, r) be a solution and define the structure group of (X, r) as the group

$$G(X, r) = \langle X \colon xy = \lambda_x(y)\rho_y(x) \rangle.$$

The solution r "extends" to a solution on $G(X, r)$.

Let (X, r) be a solution. Then there exists a unique skew brace structure on $G(X, r)$ such that

$$\begin{array}{ccc} X \times X & \xrightarrow{\quad r \quad} & X \times X \\ \downarrow{\scriptstyle \iota\times\iota} & & \downarrow{\scriptstyle \iota\times\iota} \\ G(X,r)\times G(X,r) & \xrightarrow{r_{G(X,r)}} & G(X,r)\times G(X,r) \end{array}$$

where $\iota : X \mapsto G(X, r), x \mapsto x$ is the canonical map.

If $r^2 = \mathrm{id}$ then ι è injective.

Which is the connection between being non-degenerate and being bijective?

- Any finite involutive left non-degenerate solution is non-degenerate.

  [Rump, 2005]

  [Jespers and Okniński, 2005]

  Note that $r(x, y) = (x, y)$ is involutive but clearly degenerate.

- An example of an infinite involutive solution that is left non-degenerate but not right non-degenerate. Namely, on $\mathbb{Z}$

$$
r(x,y) = \begin{cases} (y, x) & x, y \geq 0 \\ (y, x - y) & x \geq 0, y < 0 \\ (x + y, x) & x < 0, x + y \geq 0 \\ (x + y, -y) & x, x + y \geq 0. \end{cases}
$$

[Rump, 2005]

- Any non-degenerate solution such that $\lambda_x = \lambda_y$ implies $x = y$ is bijective.

[Cedó, Jespers and Verwimp, 2021]

- Any finite bijective left non-degenerate solution is right non-degenerate.

[Castelli, Catino, Stefanelli, 2021]

- Any finite left non-degenerate solution is bijective if and only if it is right non-degenerate.

[IC, Jespers, Van Antwerpen, Verwimp, 2022]

Some construction for non bijective solutions:

- One can define a skew brace like structure, called a semibrace where the additive structure is a (left cancellative) semigroup and the multiplicative structure is a group. This will give rise to (left non-degenerate) solutions.

  [Catino, IC, Stefanelli, 2017]

  [Jespers, Van Antwerpen, 2018]

- One can define a skew brace like structure where the additive structure and the multiplicative structure monoids, called YB-semitrusses. YB-semitrusses play for left non-degenerate solutions the same role as skew braces play for non-degenerate bijective solutions.

  [IC, Jespers, Van Antwerpen, Verwimp, 2022]

Connecting skew bracoids with solutions

A first attempt

We can construct a skew bracoid starting with a skew brace $(B, +, \circ)$ and a strong left ideal H. A strong left ideal I is a normal subgroup of $(B, +)$ which is λ invariant.

- For all $x \in B$ we have $x + H = x \circ H = xH$.
- The coset space B/H is a quotient group with respect to $+$, but not with respect to $\circ$.
- The group $(B, \circ)$ acts transitively on B/H by $x \odot (yH) = (x \circ y)H$.
- We have

$$\begin{aligned} x \odot (yH + zH) &= (x \circ (y + z))H \\ &= (x \circ y - x + x \circ z)H \\ &= (x \odot yH) - (x \circ eH) + (x \circ xH). \end{aligned}$$

Let $(B, +, \circ)$ be a skew brace, and suppose that there exists a strong left ideal H and a subskew brace C such that

- $(B, +) = H \ltimes C$ and
- $(B, \circ) = H \circ C$.

Consider

- the skew bracoid $(B, \circ, B/H, +, \odot)$ and
- the homomorphism $\lambda : B \to \mathrm{Aut}(B/H, +)$.

We obtain a homomorphism

$$\hat{\lambda} : B \to \mathrm{Hom}_{+}(B, C)$$

Define $\hat{\rho} : B \to \mathrm{Perm}(B)$ by

$$\hat{\lambda}_x(y) \circ \hat{\rho}_y(x) = x \circ y.$$

Theorem

The function $\hat{r} : B \times B \to B \times B$ defined by

$$\hat{r}(x, y) = (\hat{\lambda}_x(y), \hat{\rho}_y(x))$$

is aleft degenerate and right non-degenerate solution.

Questions and final remarks

- Are the solutions associated to a skew bracoid related with solutions associated to semibraces?
- Can this construction be extended to a more general setting?
- By construction $\hat{\lambda}$ is a relative gamma function on H. Can we say more?

Thank you!!!