Twisting Biquadratic Extensions

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August 4, 2023

Drinfeld Twists

Let H be a Hopf algebra.

Definition 1

[Maj00, Ex 2.3.1]
A **2-cocycle** of H is an invertible element $J \in H^{\otimes 2}$ such that

$$\Rightarrow (1 \otimes J)(1 \otimes \Delta)J = (J \otimes 1)(\Delta \otimes 1)J.$$

J is **counital** if

$$(\epsilon \otimes id)J = (id \otimes \epsilon)J = 1.$$

Definition 2

A counital 2-cocycle J of a Hopf algebra H is called a **Drinfeld twist**.

Twisting Hopf algebras

Proposition 1

[Maj00, Thm 2.3.4]
Let H be a Hopf algebra with a Drinfeld twist J. Define H^J with the same algebra structure and counit as H, and coproduct and antipode as follows;

$$\Delta_J(h) = J(\Delta(h))J^{-1}, \quad S_J(h) = U(S(h))U^{-1},$$

where $U = \Sigma J^1(S(J^2))$ is invertible.
Then H^J is a Hopf algebra.

Twisting Hopf module algebras

Proposition 2

[Maj00, Pn 2.3.8]
Let H be a Hopf algebra with a Drinfeld twist J. Let $(A, \cdot)$ be a H-module algebra. Define a product $\cdot_J : A \times A \to A$ on the set A by

$$a \cdot_J b = \cdot(J^{-1} \triangleright (a \otimes b))$$

for all $a, b \in A$.
Then $(A, \cdot_J, 1_A)$ is a new associative algebra which we label A_J.

Proposition 3

[Maj00, Pn 2.3.8]
If H is a Hopf algebra with Drinfeld twist J and A is a H-module algebra, then the twisted algebra A^J is also H^J-module algebra.

Quasitriangular Structure

Definition 3

A quasitriangular structure of H is an invertible element $\mathcal{R} \in H \otimes H$ s.t.

$$(\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, (1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12},$$
$$\tau \circ \Delta h = \mathcal{R}(\Delta h)\mathcal{R}^{-1}, \quad \forall h \in H,$$

where $\mathcal{R}_{ij} \in H^{\otimes 3}$ is $\mathcal{R}$ in the ith and jth factors and τ is the flip map defined on two vector spaces A, B as $\tau : A \otimes B \to B \otimes A, a \otimes b \mapsto b \otimes a$. A commutative quasitriangular Hopf algebra is necessarily cocommutative.

Lemma 4

$\mathcal{R}$ is a Drinfeld Twist.

Twisting a biquadratic extension I

Let $a, b \in \mathbb{Z}$ be such that $a, b > 1$ are distinct squarefree integers. Then $\mathbb{Q}(\sqrt{a}, \sqrt{b})/\mathbb{Q}$ is Galois with Galois group

$$G = V_4 = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = 1, \sigma\tau = \tau\sigma \rangle$$

where σ permutes $\sqrt{a}$ and $-\sqrt{a}$, τ permutes $\sqrt{b}$ and $-\sqrt{b}$.

$G = V_4$ is the smallest group for which there exists a cohomologically non-trivial QT structure on $\mathbb{Q}G$, given by the QT structure

$$\mathcal{R} = \frac{1}{2}(e \otimes e + \sigma \otimes e + e \otimes \tau - \sigma \otimes \tau).$$

$\mathbb{Q}V_4$ is commutative so $\mathbb{Q}V_4^{\mathcal{R}} = \mathbb{Q}V_4$. We construct a new $\mathbb{Q}V_4$-module algebra $(\mathbb{Q}(\sqrt{a}, \sqrt{b}), \star)$. By definition, $c \star d = \cdot(\mathcal{R}^{-1} \triangleright (c \otimes d))$. By a standard result $\mathcal{R}^{-1} = (S \otimes id)\mathcal{R}$. In this case, $\mathcal{R}$ is self inverse.

Twisting a biquadratic extension II

$$\begin{aligned}\sqrt{a} \star \sqrt{a} &= \cdot(\mathcal{R}^{-1} \triangleright (\sqrt{a} \otimes \sqrt{a})) \\ &= \cdot(\frac{1}{2}(e \otimes e + \sigma \otimes e + e \otimes \tau - \sigma \otimes \tau))(\sqrt{a} \otimes \sqrt{a})) \\ &= \cdot(\frac{1}{2}(\sqrt{a} \otimes \sqrt{a} - \sqrt{a} \otimes \sqrt{a} + \sqrt{a} \otimes \sqrt{a} + \sqrt{a} \otimes \sqrt{a}) = a.\end{aligned}$$

Similar computations give,

$$\sqrt{a} \star \sqrt{a} = a, \quad \sqrt{b} \star \sqrt{b} = b, \quad \sqrt{a} \star \sqrt{b} = -\sqrt{ab}, \quad \sqrt{b} \star \sqrt{a} = \sqrt{ab}.$$

Quaternion Algebras I

Definition 5

[Voi23] An algebra B over F is a quaternion algebra if there exist $i, j \in B$ such that $1, i, j, ij$ is an F-basis for B and

$$i^2 = a, j^2 = b, \text{ and } ji = ij$$

for some $a, b \in F^\times$. We write $B = \left(\frac{a,b}{F}\right)$.

Returning to our previous example, define a map of algebras

$$T : (\mathbb{Q}(\sqrt{a}, \sqrt{b}), \star) \to \left(\frac{a, b}{\mathbb{Q}}\right),$$

$$T(\sqrt{a}) = i, T(\sqrt{b}) = j, T(\sqrt{a}b) = k$$

Quaternion Algebras II

Then $(\mathbb{Q}(\sqrt{a}, \sqrt{b}), \star) \cong \left(\frac{a,b}{\mathbb{Q}}\right)$, so it is either a division algebra or the matrix algebra $M_2(\mathbb{Q})$.
A quaternion algebra is a division algebra iff $N(q) \neq 0$, $\forall 0 \neq q = \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 k$.

Let b be odd and suppose that the Jacobi symbol $\left(\frac{a}{b}\right) = -1$.
We show that $\left(\frac{a,b}{\mathbb{Q}}\right)$ is a division algebra.

Suppose $\exists$ a rational, non-trivial solution to $\lambda_1^2 - \lambda_2^2 a - \lambda_3^2 b + \lambda_4^2 ab = 0$ given by $(\alpha, \beta, \gamma, \delta)$ where $\alpha, \beta, \gamma, \delta \in \mathbb{Q}$ not all zero. Then $\exists$ an integer solution (k, l, m, n), not all zero such that the non zero elements are coprime.

Quaternion Algebras III

Since $\left(\frac{a}{b}\right) = -1$, $\exists$ a prime p dividing b such that a is a quadratic non-residue mod p. Let $b = pb'$ where $p \nmid b'$ and consider

$$\begin{aligned} k^2 - al^2 - bm^2 + abn^2 &= 0 \\ \Rightarrow k^2 &\equiv al^2 \mod p \\ \Rightarrow k, l &\equiv 0 \mod p \\ \Rightarrow k^2 - al^2 &\equiv 0 \mod p^2 \\ \Rightarrow pb'm^2 + apb'n^2 &\equiv 0 \mod p^2 \\ \Rightarrow b'm^2 + ab'n^2 &\equiv 0 \mod p \\ \Rightarrow m, n &\equiv 0 \mod p. \end{aligned}$$

This contradicts the assumption that the solution is coprime. Thus $\left(\frac{a,b}{\mathbb{Q}}\right)$ is a division algebra.

H-Galois extension of fields

Definition 6

[Chi00, Definition 2.7]
Given a K Hopf algebra H, a field extension L/K is said to be a H-Galois extension if L is a H-module algebra and the map

$$j : L\#H \to End_K(L),$$
$$j(s \otimes_K h)(x) = sh(x) \quad \forall s, x \in L, h \in H$$

is an isomorphism of K-vector spaces.

H-Galois extension of algebras

Definition 7

[KT81] Let R be a commutative ring, B an R-algebra with subalgebra $A \subseteq B$. Let H be a finitely generated R-Hopf algebra coacting on B. Then B is a H-Galois extension of A if

1. B is a right H-comodule algebra,
2. $A = \{a \in B | \phi(a) = u^*(\phi).a, \quad \forall \phi \in H^*\}$. That is $A = B^{CoH}$.
3. The left B-module homomorphism $\lambda : B \otimes_A B \to B \otimes_R H$ given by $\lambda(x \otimes y) = xy^{(1)} \otimes y^{(2)}$ is surjective.

Example 8

Let L/K be a finite extension of fields. Then set $R = K, A = K, B = L, H = (KG)^*$ in the previous definition.
If L/K is a Galois extension of fields, then it is a H-Galois extension of algebras. Moreover, the map λ is the dual map of j.

By a theorem 2.5 stated in [EN22], if a f.d. Hopf algebra H acts on a division algebra $\mathcal{Q}$ then $\mathcal{Q}/\mathcal{Q}^H$ is H-Galois iff $rank_{\mathcal{Q}^H}\mathcal{Q} = dimH$.

We have $H = \mathbb{Q}V_4$ acting on $\left(\frac{a,b}{\mathbb{Q}}\right)$.

$$\sigma \in V_4, \sigma(m) = m \iff m \in \mathbb{Q}$$
$$\Rightarrow \left(\frac{a,b}{\mathbb{Q}}\right)^H = \mathbb{Q}.$$

Moreover, $rank_{\left(\frac{a,b}{\mathbb{Q}}\right)^H}\left(\frac{a,b}{\mathbb{Q}}\right) = rank_{\mathbb{Q}}\left(\frac{a,b}{\mathbb{Q}}\right) = 4 = dimH$.

Thus $\left(\frac{a,b}{\mathbb{Q}}\right)/\mathbb{Q}$ is a $\mathbb{Q}V_4$-Galois extension.

Questions...

What can the arithmetic in the biquadratic field extension tell us about the arithmetic in the quaternion algebra?

For example, in a biquadratic field extension $L = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ the ring of algebraic integers O_L is the maximal order in L.

- What happens to O_L under the twisting operation?
  - Is it closed under the $\star$-product?
  - If so, is the new order also maximal?

Further questions...

- Twisting process is invertible.

- Given a quaternion algebra $L = \left(\frac{a,a}{K}\right)$ it's clear that V_4 acts covariantly on L. So L is a KV_4-module algebra.

- If we apply the twisting process to this structure what is the resulting twisted algebra?

Bibliography

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