Homomorphisms and Short Exact Sequences of Skew Bracoids

Isabel Martin-Lyons

joint work with Paul Truman

Keele University, UK

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Outline

1. Basic Definitions and Substructures
2. Homomorphisms and Isomorphisms
3. Images and Kernels
4. A Motivating Example

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The Objects at Play

Definition

A *skew (left) brace* is a triple $(G, \star, \circ)$, where $(G, \star)$ and $(G, \circ)$ are groups and for all $g, h, f \in G$

$$g \circ (h \star f) = (g \circ h) \star g^{-1} \star (g \circ f).$$

Definition

A *skew (left) bracoid* is a 5-tuple $(G, \cdot, N, \star, \odot)$, where $(G, \cdot)$ and $(N, \star)$ are groups and $\odot$ is a transitive action of G on N for which

$$g \odot (\eta \star \mu) = (g \odot \eta) \star (g \odot e_N)^{-1} \star (g \odot \mu),$$

for all $g \in G$ and $\eta, \mu \in N$.

Housekeeping

- We will assume everything is finite.
- We will frequently write $(G, N, \odot)$, or even (G, N), for $(G, \cdot, N, \star, \odot)$.
- We will refer to $(N, \star)$ as the additive group and $(G, \cdot)$ as the multiplicative or acting group.
- Any identity will be denoted e, possibly with a subscript.

For example

Examples

- Any skew brace $(G, \star, \circ)$ can be thought of as a skew bracoid $(G, \circ, G, \star, \odot)$, where $\odot$ is simply $\circ$.

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- For any group (G) we have the skew bracoid $(G, \{e\}, \odot)$ where of course the action $\odot$ is trivial.
- Let $d, n \in \mathbb{N}$ such that $d|n$. Take $G = \langle r, s \mid r^n = s^2 = e, srs^{-1} = r^{-1} \rangle \cong D_n$ and $N = \langle \eta \rangle \cong C_d$. Then we get a skew bracoid $(G, N, \odot)$ for $\odot$ given by

$$r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}.$$

The γ-functions

Definition/Proposition

Let $(G, N, \odot)$ be a skew bracoid and $g \in G$. The map $\gamma_g : N \to N$ given by

$$\gamma_g(\eta) = (g \odot e_N)^{-1}(g \odot \eta)$$

is in fact an automorphism of N.

We call these maps associated with the skew bracoid the γ-*functions* of the skew bracoid.

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We can use these maps to embed G into $\mathrm{Hol}(N)$ via $g \mapsto (g \odot e_N, \gamma_g)$. Morally, this amounts to killing off any kernel of the action $\odot$.

Substructures

Let $(G, N, \odot)$ be a skew bracoid.

Definition

The triple $(H, M, \odot)$ is a *sub-skew bracoid* of $(G, N, \odot)$ if and only if

- H is a subgroup of G,
- M is a subgroup of N,
- and H acts transitively on M via $\odot$.

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Definition

A normal subgroup M of N is an *ideal* of $(G, N, \odot)$ if and only if it is closed under γ_g for all $g \in G$.

Quotients

Proposition

Let $(G, N, \odot)$ be a skew bracoid with M an ideal. We have that G acts on the quotient group N/M via $g \odot (\eta M) := (g \odot \eta)M$, and $(G, N/M, \odot)$ is a skew bracoid.

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Homomorphisms

Definition

A *homomorphism* of skew bracoids between $(G, N, \odot)$ and $(G', N', \odot')$ is a pair of group homomorphisms $\varphi : G \to G'$ and $\psi : N \to N'$ such that

$$\psi(g \odot \eta) = \varphi(g) \odot' \psi(\eta)$$

for all $g \in G$ and $\eta \in N$.

An alternative framing

Let $\varphi : G \to G'$ be a homomorphism of groups.

If $\varphi(\mathrm{Stab}_G(e_N)) \subseteq \mathrm{Stab}_{G'}(e_{N'})$ then we have a well-defined map $\varphi_N : N \to N'$ given by

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If this φ_N turns out to be a homomorphism itself then the pair, φ and φ_N, forms a homomorphism of skew bracoids.

Conversely every homomorphism of skew bracoids is necessarily of this form.

Notation

With this in mind, we set the following convention:

- $\boldsymbol{\varphi}$ denotes the pair of homomorphisms that form the skew bracoid homomorphism,
- φ denotes the homomorphism between the acting groups,
- and φ_N the (induced) homomorphism between the additive groups.

Isomorphism and Equivalence

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Definition

We say that two skew bracoids $(G, N, \odot)$ and $(G', N', \odot')$ are equivalent if and only if $N = N'$ and the image of G and the image of G' in $\mathrm{Hol}(N)$ coincide.

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This has the uncomfortable consequence that a homomorphism can be injective and surjective, but not an isomorphism. We can take comfort in the fact that it will be an isomorphism, up to our notion of equivalence.

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Images and Kernels

Proposition

Let $\varphi : (G, N, \odot) \to (G', N', \odot')$ be a homomorphism of skew bracoids then

- $\ker(\varphi_N)$ is an ideal of $(G, N, \odot)$,
- and $(\mathrm{im}(\varphi), \mathrm{im}(\varphi_N), \odot')$ is a sub-skew bracoid of $(G', N', \odot')$.

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Definition

The *image* of the homomorphism φ is $\mathrm{im}(\varphi) := (\mathrm{im}(\varphi), \mathrm{im}(\varphi_N))$.

Kernel For Our Purposes

Remember our goal is to develop a notion of short exact sequence for skew bracoids. For there to be any hope of the kernel of a homomorphism and the image of a homomorphism aligning, we need them to be the same kind of object.

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Naive Attempt: $(\ker(\varphi), \ker(\varphi_N))$

Consider the (toy) example $\boldsymbol{\varphi} : (D_3, C_3) \to (D_3, \{e\})$ where φ is just the identity, and $\varphi_N(\eta) = e$ for all $\eta \in C_3$.

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But $\ker(\varphi) = \{e\}$ and $\ker(\varphi_N) = C_3$, so $\ker(\varphi)$ does not act transitively on $\ker(\varphi_N)$ and thus we so not have a skew bracoid.

For Our Purposes

Proposition (Intelligent Attempt)

Let $\varphi : (G, N, \odot) \to (G', N', \odot')$ be a homomorphism of skew bracoids and $S' = \mathrm{Stab}_{G'}(e_{N'})$. The triple $\ker(\varphi) := (\varphi^{-1}(S'), \ker(\varphi_N), \odot)$ is a sub-skew bracoid of $(G, N, \odot)$.

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- In our toy example $S' = D_3$ and $\varphi^{-1}(D_3) = D_3$ so that our "kernel" is (D_3, C_3). Certainly a skew bracoid.
- Specialising to the skew brace case, this stabiliser will necessarily be trivial so its inverse image is precisely the kernel of φ.
- This lands on the associated subgroup in G of $\ker(\varphi_N)$, our standard lift from an ideal in the additive group to subgroup of the acting group.

Short Exact Sequence

Then a short exact sequence of skew bracoids is given by

$$e \longrightarrow (G_1, N_1) \overset{\varphi}{\hookrightarrow} (G_2, N_2) \overset{\psi}{\twoheadrightarrow} (G_3, N_3) \longrightarrow e$$

where φ and ψ are homomorphisms of skew bracoids such that

$$\mathrm{im}(\varphi) = \ker(\psi).$$

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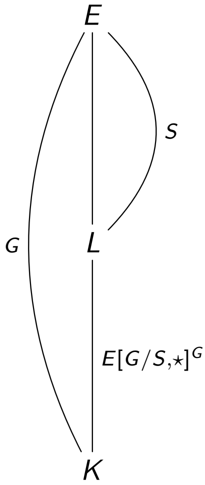
The Correspondence between Hopf-Galois Structures and Skew Bracoids

Let L/K be a separable extension of fields with Galois closure E, and write $(G, \cdot) = \mathrm{Gal}(E/K)$ and $S = \mathrm{Gal}(E/L)$. Recall



Theorem

There is a bijective correspondence between

- Hopf-Galois structures on L/K and
- operations $\star$ such that $(G, \cdot, G/S, \star, \odot)$ forms a skew bracoid, where $\odot$ is left translation of cosets via $\cdot$.

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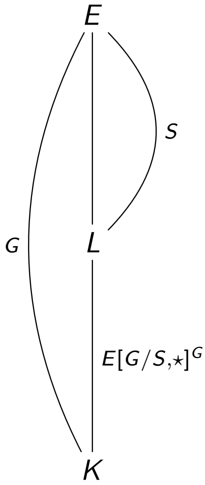

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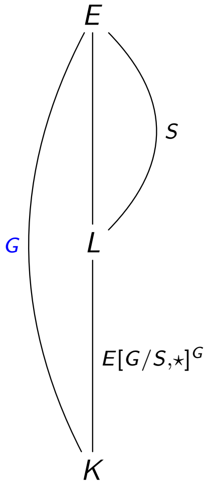

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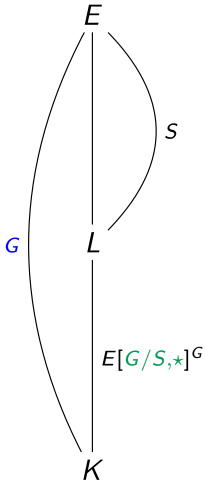

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Definition

A Hopf-Galois structure is *almost classical* if it corresponds under Greither-Pareigis to a subgroup of $\mathrm{Perm}(G/S)$ of the form $\lambda(H)^{opp}$, for some normal complement H of S.

Almost Classical Skew Bracoids

Definition

Let (G, N) be a skew bracoid and $S = \mathrm{Stab}_G(e_N)$. We say (G, N) is *almost classical* if S has a normal complement H in G for which (H, N) is essentially a trivial skew brace.

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This means that for all $h_1, h_2 \in H$ we have

$$h_1 h_2 \odot e_N = (h_1 \odot e_N)(h_2 \odot e_N).$$

For Example

Example

We can take $(G, N, \odot)$ with $G = \langle r, s \mid r^3 = s^2 = e, srs^{-1} = r^{-1} \rangle \cong D_3$, $N = \langle \eta \rangle \cong C_3$, with $\odot$ given by

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- Putting $k = 0$ in (1), we get $r^i s^j \odot e_N = \eta^i$. We stabilise e_N precisely when $i = 0$, so $\mathrm{Stab}_G(e_N) = \langle s \rangle$.

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- For $0 \leq i, j \leq 2$, we have $(r^i \odot e)(r^j \odot e) = \eta^{i+j} = r^i r^j \odot e$.

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- For $0 \leq i, j \leq 2$, we have $(r^i \odot e)(r^j \odot e) = \eta^{i+j} = r^i r^j \odot e$.
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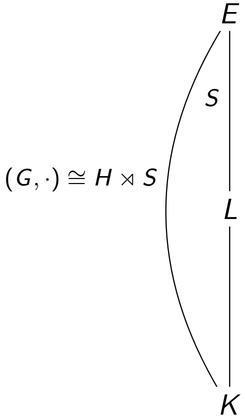
Hence we have an almost classical skew bracoid.

Induced Hopf-Galois Structures



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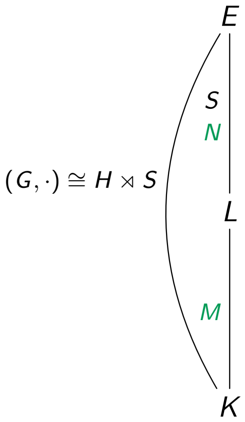

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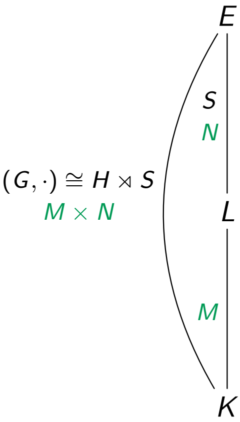

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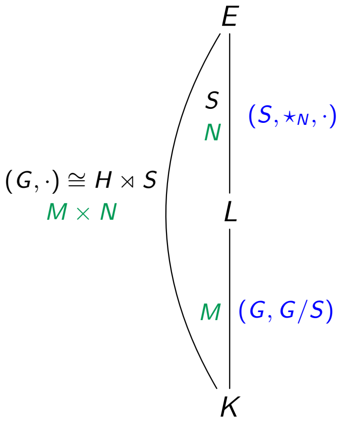

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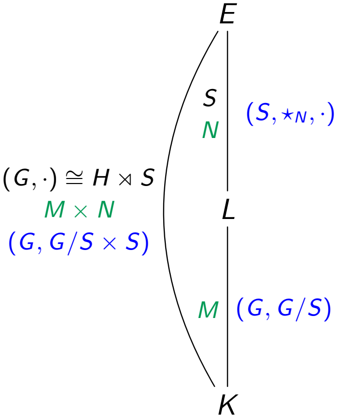

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Taking a direct product of the additive groups of $(G, \cdot, G/S, \star_M, \odot)$ and $(S, \star_N, \cdot)$ we can construct a skew bracoid $(G, G/S \times S)$ on E/K, which is essentially a skew brace.

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We have the trivial skew brace $(S, \cdot, \cdot)$ and the almost classical skew bracoid $(G, G/S)$ with $G/S \cong \langle r \rangle$. Taking the direct product of G/S and S we get a cyclic group of order 6 generated by (rS, s) for example. Associating (rS, e) with r and (eS, s) with s, we get a skew brace $(G, \star, \cdot)$ where $(G, \star) \cong C_6$ and $(G, \cdot) \cong D_3$.

The Short Exact Sequence

With this machinery in tow we have the following.

Proposition

Let (G, H) be an almost classical skew bracoid and $S = \mathrm{Stab}_G(e_H)$. We have the short exact sequence

$$e \longrightarrow (S, S) \hookrightarrow (G, H \times S) \twoheadrightarrow (G, H) \longrightarrow e.$$

Thank you for your attention!