Hopf-Galois Structures on Parallel Extensions

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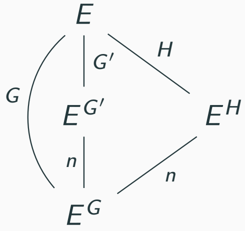

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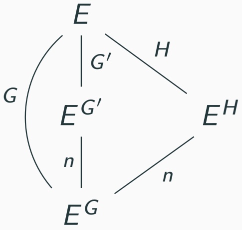

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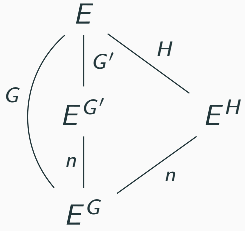

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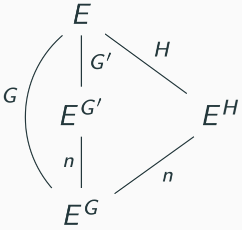

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- There may be other index n subgroups H of G.
- We say L'/K is a *parallel* extension of L/K.

Question...

If L/K admits a Hopf-Galois structure of type N, what can we say about L'/K?

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$$\sigma(\sqrt[4]{2}) = -i\sqrt[4]{2}, \qquad \sigma(i) = i,$$
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- $L = E^{G'}$ where $G' = \langle \tau \rangle$ is index 4 in G and has trivial core.
- $H := \langle \sigma^2 \rangle$ is a normal subgroup in G, so $E^H = \mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$ is Galois with group $C_2 \times C_2$ (smaller Galois closure, so possibly different HGS).

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We saw an example where $G' \cong H$ as abstract groups, but $L \ncong L'$ as field extensions. **Isomorphism need not preserve normality**.

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Answer: Suppose $C := \mathrm{Core}_G(H) \neq \{1\}, H$. Then L'/K has normal closure E^C. If there is no N of order $n := [L : K] = [L' : K]$ s.t. $G/C \cong \mathrm{Gal}(E^C/K)$ is isomorphic to a regular subgroup of $\mathrm{Hol}(N)$, then L'/K has no HGS.

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Question: can this work without a full classification beforehand?

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For any a, b, we have $\mathrm{Core}_G(H_1) = \{1\}$, and if $\mathrm{ord}(\alpha) > q$, then $\mathrm{Core}_G(H_2) = \langle\tau\rangle$.

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Note: $\langle\sigma\rangle \rtimes \langle\alpha\rangle$ is transitive on C_p, but $[L' : K] = pq$, so L'/K admits no HGS.

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- $\#\mathrm{Aut}(G)$-orbits $= \#\mathrm{Aut}(G \cap Q)$-orbits

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- How ‘spread out’ are parallel extensions admitting no HGS?
- Does this make sense for skew bracoids?