# Extensions of local fields in characteristic 0 and characteristic $p$

## (d'après P. Deligne)

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June 1, 2023

# Notation

Let $K$ be a local field, complete w.r.t. the valuation $v_K : K^\times \twoheadrightarrow \mathbb{Z}$. Then $v_K$ extends uniquely to a valuation on the separable closure $K^{sep}$ of $K$, which we also denote by $v_K$. Define

$$\mathcal{O}_K = \{x \in K : v_K(x) \geq 0\}$$
$$\mathcal{P}_K = \{x \in K : v_K(x) > 0\}.$$

Assume that the residue field $k = \mathcal{O}_K/\mathcal{P}_K$ of $K$ is a perfect field of characteristic $p$.

If $\mathrm{char}(K) = 0$ and $k$ is finite, then $K$ is a finite extension of $\mathbb{Q}_p$.
If $\mathrm{char}(K) = p$ then $K \cong k((t))$.

Let $e_K = v_K(p)$ be the absolute ramification index of $K$.

Let $\pi_K$ be a uniformizer for $K$. Thus $v_K(\pi_K) = 1$.

If $L/K$ is an extension of local fields let $e_{L/K} = v_L(\pi_K)$ denote the ramification index of $L/K$.

# A not-so-crazy idea

If $\mathrm{char}(K) = 0$ but $e_K$ is large then $K$ should be similar to a local field of characteristic $p$.

In fact, let $K$ be a local field of characteristic 0 with residue field $k$, and let $F$ be a local field of characteristic $p$ with the same residue field $k$. Then for $1 \leq d \leq e_K$ we have

$$\mathcal{O}_K/\mathcal{P}_K^d \cong k[[t]]/(t^d) \cong \mathcal{O}_F/\mathcal{P}_F^d.$$

Deligne [De79] devised a theory of extensions of "truncated valuation rings" which correspond to extensions of both $K$ and $F$, but with bounds on ramification.

This leads to a one-to-one correspondence between extensions of $K$ with bounded ramification and extensions of $F$ with bounded ramification.

# The category of truncated valuation rings

Define a category $\mathcal{T}$ whose objects are triples $(A, M, \epsilon)$, where

- $A$ is an Artin local ring whose maximal ideal $\mathcal{P}_A$ is principal.
- $A/\mathcal{P}_A$ is a perfect field with positive characteristic.
- $M$ is a free $A$-module of rank 1.
- $\epsilon : M \to A$ is an $A$-module homomorphism such that $\epsilon(M) = \mathcal{P}_A$.

Define the length of $S = (A, M, \epsilon)$ to be the length of the $A$ as an $A$-module.

**Example**

Let $K$ be a local field and let $d \in \mathbb{N}$. Set $A = \mathcal{O}_K/\mathcal{P}_K^d$, $M = \mathcal{P}_K/\mathcal{P}_K^{d+1}$, and let $\epsilon : \mathcal{P}_K/\mathcal{P}_K^{d+1} \to \mathcal{O}_K/\mathcal{P}_K^d$ be induced by the inclusion $\mathcal{P}_K \hookrightarrow \mathcal{O}_K$. Then $(A, M, \epsilon)$ is an element of $\mathcal{T}$ with length $d$.

Denote the triple constructed in the example by $\mathrm{Tr}_d(K)$.

## Morphisms in $\mathcal{T}$

For $i = 1, 2$ let $S_i = (A_i, M_i, \epsilon_i) \in \mathcal{T}$. A $\mathcal{T}$-morphism $f : S_1 \to S_2$ is a triple $(e, \mu, \eta)$ where

- $e \geq 1$. (Say $e$ is the ramification index of $S_2$ over $S_1$.)
- $\mu : A_1 \to A_2$ is a ring homomorphism.
- $\eta : M_1 \to M_2^{\otimes e}$ is an $A_1$-module homomorphism.

(Here $M_2^{\otimes e} = M_2 \otimes_{A_2} \cdots \otimes_{A_2} M_2$, with $e$ factors.)

These must satisfy

- $\mu \circ \epsilon_1 = \epsilon_2^e \circ \eta$

$$\begin{array}{ccc} M_1 & \xrightarrow{\eta} & M_2^{\otimes e} \\ {\scriptstyle \epsilon_1}\downarrow & & \downarrow{\scriptstyle \epsilon_2^e} \\ A_1 & \xrightarrow[\mu]{} & A_2 \end{array}$$

- $\eta$ induces an $A_2$-module isomorphism $M_1 \otimes_{A_1} A_2 \cong M_2^{\otimes e}$.

Let $f = (e, \mu, \eta)$ be a $\mathcal{T}$-morphism from $S_1$ to $S_2$. Say $f$ is a $\mathcal{T}$-extension if $\mathrm{length}(S_2) = e \cdot \mathrm{length}(S_1)$. In this case we write $S_2/S_1$.

# Morphisms in $\mathcal{T}$ (continued)

If we have another $\mathcal{T}$-morphism $g : S_2 \to S_3$ given by $g = (e', \nu, \theta)$ then $g \circ f : S_1 \to S_3$ is defined by $g \circ f = (e e', \nu \circ \mu, \theta^{\otimes e} \circ \eta)$.

We have $\mathrm{id}_{S_i} = (1, \mathrm{id}_{A_i}, \mathrm{id}_{M_i})$. Let $f = (e, \mu, \eta)$ be a $\mathcal{T}$-morphism from $S_1$ to $S_2$. Then $f$ is an isomorphism if and only if $e = 1$, $\mu$ is a ring isomorphism, and $\eta$ is an isomorphism of $A_1$-modules.

### Proposition ([De79, 1.2])

*Let $S = (A, M, \epsilon) \in \mathcal{T}$. Then there is a local field $K$ and $d \in \mathbb{N}$ such that $S \cong Tr_d(K)$.*

### Example

If $1 \le d \le e_K$ then $\mathrm{Tr}_d(K) \cong (k[[t]]/(t^d), (t)/(t^{d+1}), \epsilon)$, where $k = \mathcal{O}_K/\mathcal{P}_K$ and $\epsilon$ is induced by $(t) \hookrightarrow k[[t]]$.

# $\mathcal{T}$-extensions induced by field embeddings

Let $K$, $L$ be local fields and let $\sigma : K \to L$ be a field embedding. Then for $d \in \mathbb{N}$ we get a $\mathcal{T}$-morphism $f_\sigma = (e_\sigma, \mu_\sigma, \eta_\sigma)$ from $\mathrm{Tr}_d(K)$ to $\mathrm{Tr}_{de_\sigma}(L)$, where

- $e_\sigma$ is the ramification index of $L$ over $\sigma(K)$.
- $\mu_\sigma : \mathcal{O}_K/\mathcal{P}_K^d \to \mathcal{O}_L/\mathcal{P}_L^{de_\sigma}$ is induced by $\sigma$.
- $\eta_\sigma : \mathcal{P}_K/\mathcal{P}_K^{d+1} \to \mathcal{P}_L^{e_\sigma}/\mathcal{P}_L^{e_\sigma(d+1)} \cong (\mathcal{P}_L/\mathcal{P}_L^{e_\sigma d+1})^{\otimes e_\sigma}$ is induced by $\sigma$.

It follows from the definitions that $f_\sigma : \mathrm{Tr}_d(K) \to \mathrm{Tr}_{de_\sigma}(L)$ is an extension of truncated valuation rings.

# The category of extensions of a truncated valuation ring

Let $S_1 = (A_1, M_1, \epsilon_1)$ be an object in $\mathcal{T}$. Define a category $\mathrm{ext}(S_1)$ whose objects are $\mathcal{T}$-extensions $f : S_1 \to S_2$. If $g : S_1 \to S_3$ is another object in $\mathrm{ext}(S_1)$, a morphism from $S_2/S_1$ to $S_3/S_1$ is a $\mathcal{T}$-morphism $h : S_2 \to S_3$ such that $h \circ f = g$.

$$\begin{tikzcd} S_2 \arrow[rr, "h"] & & S_3 \\ & S_1 \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

**Proposition ([De79], Lemma 1.4.4)**

*Let $S/Tr_d(K)$ be a $\mathcal{T}$-extension with ramification index $e$. Then there is a finite separable field extension $L/K$ such that $Tr_{de}(L)/Tr_d(K)$ is isomorphic to $S/Tr_d(K)$ in the category $\mathrm{ext}(Tr_d(K))$.*

# A functor

Let $K$ be a local field, and let $\text{ext}(K)$ be the category of finite separable extensions of $K$.

Let $d \in \mathbb{N}$. We get a functor $\mathcal{F}_d : \text{ext}(K) \to \text{ext}(\text{Tr}_d(K))$ by $\mathcal{F}_d(L) = \text{Tr}_{de_{L/K}}(L)$. For a $K$-embedding $\sigma : L \to L'$ we define $\mathcal{F}_d(\sigma) = f_\sigma$, where

$$f_\sigma : \text{Tr}_{de_{L/K}}(L) \longrightarrow \text{Tr}_{de_{L'/K}}(L')$$

was defined above.

The functor $\mathcal{F}_d$ is essentially surjective, but it need not be full or faithful ...

# An example

Let $K = k((t))$ be a local field of characteristic $p$, and let $L$ be the extension of $K$ generated by a root of $g(X) = X^p - X - t^{-2p+1}$. Then $L/K$ is a totally ramified cyclic extension of degree $p$, with $e_{L/K} = p$.

Let $\alpha$ be a root of $g(X)$. Then there is $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\alpha) = \alpha + 1$. Set $\pi_L = t^2\alpha$. Then $v_L(\pi_L) = 1$ and $\sigma(\pi_L) = \pi_L + t^2$.

Taking $d = 1$, we find that $\mathrm{Tr}_p(L)$ is a $\mathcal{T}$-extension of $\mathrm{Tr}_1(K)$. The functor

$$\mathcal{F}_1 : \mathrm{ext}(K) \longrightarrow \mathrm{ext}(\mathrm{Tr}_1(K))$$

gives a homomorphism from $\mathrm{Gal}(L/K)$ to $\mathrm{Aut}(\mathrm{Tr}_p(L)/\mathrm{Tr}_1(K))$.

Since $\sigma$ induces the identity on both $\mathcal{O}_L/\mathcal{P}_L^p$ and $\mathcal{P}_L/\mathcal{P}_L^{p+1}$, we have $\mathcal{F}_1(\sigma) = \mathcal{F}_1(\mathrm{id}_L)$. Hence $\mathcal{F}_1$ is not faithful.

# Example (continued)

On the other hand, for $c \in k^\times$ we can define an $\mathrm{ext}(\mathrm{Tr}_1(K))$-morphism $g_c : \mathrm{Tr}_p(L) \to \mathrm{Tr}_p(L)$ by $g_c = (1, \mathrm{id}_{\mathcal{O}_L/\mathcal{P}_L^p}, \eta_c)$, where

$$\eta_c : \mathcal{P}_L/\mathcal{P}_L^{p+1} \longrightarrow \mathcal{P}_L/\mathcal{P}_L^{p+1}$$

is the $\mathcal{O}_L/\mathcal{P}_L^p$-module homomorphism defined by

$$\eta_c(\pi_L + \mathcal{P}_L^{p+1}) = \pi_L + c\pi_L^p + \mathcal{P}_L^{p+1}.$$

So $\mathcal{F}_1$ is not full.

Similar arguments show that

$$\mathcal{F}_d : \mathrm{ext}(K) \longrightarrow \mathrm{ext}(\mathrm{Tr}_d(K))$$

is neither full nor faithful for any $d \in \mathbb{N}$.

# Higher ramification theory

Let $L/K$ be a separable extension of local fields. Let $L_0/K$ be the maximal unramified subextension of $L/K$ and set

$$\Gamma = \left\{\sigma : L \to K^{sep} \,\middle|\, \sigma|_K = \mathrm{id}_K\right\}$$
$$\Gamma_0 = \{\sigma \in \Gamma : \sigma|_{L_0} = \mathrm{id}_{L_0}\}.$$

For nonnegative real $x$ define a subset of $\Gamma$ by

$$\Gamma_x = \{\sigma \in \Gamma_0 : v_L(\sigma(\pi_L) - \pi_L) \geq x + 1\}.$$

Say $\Gamma_x$ is the $x$th lower ramification subset of $\Gamma$.

We have the following:

- If $x \geq y \geq 0$ then $\Gamma_x \subset \Gamma_y$.
- $\Gamma_x = \{\mathrm{id}_L\}$ for all sufficiently large $x$.

# The upper numbering for ramification sets

Define the Hasse-Herbrand function $\phi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\phi_{L/K}(x) = \frac{1}{[L : L_0]} \int_0^x |\Gamma_t| \, dt.$$

Then $\phi_{L/K}$ is a continuous bijection, so we can define $\psi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $\psi_{L/K} = \phi_{L/K}^{-1}$.

For $y \geq 0$ set $\Gamma^y = \Gamma_{\psi_{L/K}(y)}$. Say $\Gamma^y$ is the $y$th upper ramification subset of $\Gamma$.

We have $\Gamma^y = \{\mathrm{id}_L\}$ for $y$ sufficiently large. Set

$$m_{L/K} = \inf\{y \geq 0 : \Gamma^y = \{\mathrm{id}_L\}\}.$$

# An example

Let $K = \mathbb{Q}_2$ and let $L$ be generated over $\mathbb{Q}_2$ by a root $\pi_L = 2^{1/4}$ of $g(X) = X^4 - 2$. Then $L = \mathbb{Q}_2(\pi_L)$.

Let $i$ be a primitive 4th root of unity. Then the $\mathbb{Q}_2$-embeddings of $L$ into $\mathbb{Q}_2^{sep}$ are defined by

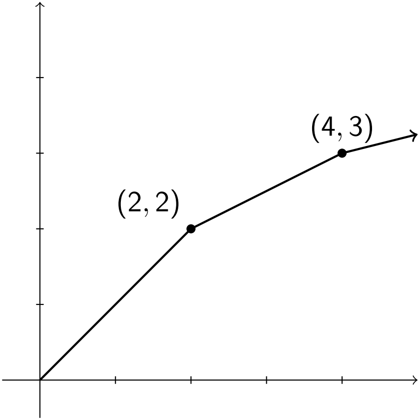
$$\begin{array}{cc} \sigma_1(\pi_L) = \pi_L & \sigma_i(\pi_L) = \pi_L i \\ \sigma_{-1}(\pi_L) = -\pi_L & \sigma_{-i}(\pi_L) = -\pi_L i. \end{array}$$

We have $v_L(\sigma_{-1}(\pi_L) - \pi_L) = 5$ and $v_L(\sigma_{\pm i}(\pi_L) - \pi_L) = 3$. Hence

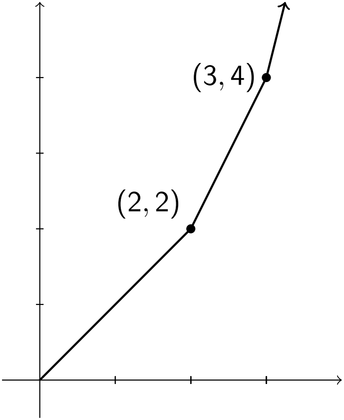
$$|\Gamma_x| = \begin{cases} 4 & (0 \le x \le 2) \\ 2 & (2 < x \le 4) \\ 1 & (4 < x). \end{cases} \qquad |\Gamma^y| = \begin{cases} 4 & (0 \le y \le 2) \\ 2 & (2 < y \le 3) \\ 1 & (3 < y). \end{cases}$$

It follows that $m_{L/K} = 3$.

# Hasse-Herbrand functions for $L/K$




$y = \phi_{L/K}(x)$




$y = \psi_{L/K}(x)$

# A condition on field extensions

Let $L/K$ be a finite separable extension of local fields and let $a \geq 0$. Say $L/K$ satisfies condition $C^a$ if one of the following equivalent conditions holds:

- $\Gamma^a = \{\mathrm{id}_L\}$
- $a > m_{L/K}$.

For a local field $K$ let $\mathrm{ext}(K)^a$ denote the full subcategory of $\mathrm{ext}(K)$ whose objects are finite separable extensions of $K$ satisfying $C^a$.

# Condition $C^a$ for truncated valuation rings

Let $S \in \mathcal{T}$ and set $d = \mathrm{length}(S)$. Then there is a local field $K$ such that $S \cong \mathrm{Tr}_d(K)$.

Let $T/S \in \mathrm{ext}(S)$, and let $e$ be the ramification index of $T$ over $S$. Then there is a separable extension $L/K$ such that $\mathrm{Tr}_{de}(L) \cong T$ and following diagram commutes:

$$\begin{tikzcd} \mathrm{Tr}_{de}(L) \arrow[r, "\sim"] & T \\ \mathrm{Tr}_d(K) \arrow[r, "\sim"] \arrow[u] & S \arrow[u] \end{tikzcd}$$

Let $a \leq d$. We say that the $\mathcal{T}$-extension $T/S$ satisfies $C^a$ if $L/K$ satisfies $C^a$. This definition depends only on the $\mathcal{T}$-extension $T/S$, and not on the choices of $K$ and $L$.

We define a category $\mathrm{ext}(S)^a$ whose objects are $\mathcal{T}$-extensions of $S$ which satisfy $C^a$.

# An equivalence relation on $\mathcal{T}$-morphisms

Let $c \in \mathbb{N}$ and let $f = (e, \mu, \eta)$, $f' = (e', \mu', \eta')$ be $\mathcal{T}$-morphisms from $S_1 = (A_1, M_1, \epsilon_1)$ to $S_2 = (A_2, M_2, \epsilon_2)$. Say $f \equiv f' \pmod{R(c)}$ if all of the following hold:

- $e = e'$.
- $\mu$ and $\mu'$ induce the same map from $A_1/M_1$ to $A_2/M_2$.
- $\eta(x) - \eta'(x) \in \mathcal{P}^{ec}_{A_2} M_2^{\otimes e}$ for all $x \in M_1$.

Suppose $S_1 = \mathrm{Tr}_d(L_1)$, $S_2 = \mathrm{Tr}_{de}(L_2)$, $f = f_\sigma$, and $f' = f_{\sigma'}$ for some embeddings $\sigma, \sigma' : L_1 \to L_2$. Then the second and third conditions are equivalent to

$$v_{L_2}(\sigma(\alpha) - \sigma'(\alpha)) \geq e(c+1)$$

for all $\alpha \in \mathcal{O}_{L_1}$.

# An equivalence of categories

Let $T/S$ and $T'/S$ be objects in $\mathrm{ext}(S)^a$. Then there are $d \in \mathbb{N}$, a local field $K$, and extensions $L/K$, $L'/K$ such that

$$T/S \cong \mathrm{Tr}_{de_{L/K}}(L)/\mathrm{Tr}_d(K)$$
$$T'/S \cong \mathrm{Tr}_{de_{L'/K}}(L')/\mathrm{Tr}_d(K).$$

A morphism from $T/S$ to $T'/S$ in $\mathrm{ext}(S)^a$ is defined to be an $R(\psi_{L/K}(a))$-equivalence class of morphisms in $\mathrm{ext}(S)$.

**Theorem ([De79], Theorem 2.8)**

*Let $d \in \mathbb{N}$ and $1 \leq a \leq d$. Then the functor $\mathcal{F} : ext(K)^a \to ext(Tr_d(K))^a$ which maps $L/K$ to $Tr_{de_{L/K}}(L)/Tr_d(K)$ is an equivalence of categories.*

# An application

**Corollary**

*Let $K$ and $F$ be local fields which have the same residue field $k$ and satisfy $e_K \leq e_F$. Then for $1 \leq d \leq e_K$ there is an equivalence of categories between $\mathrm{ext}(K)^d$ and $\mathrm{ext}(F)^d$.*

*In particular, there is a one-to-one correspondence between separable extensions $L/K$ with $m_{L/K} < e_K$ and separable extensions $E/F$ with $m_{E/F} < e_K$.*

Proof: Since $d \leq e_K \leq e_F$ we have

$$\mathrm{Tr}_d(K) \cong (k[[t]]/(t^d), (t)/(t^{d+1}), \epsilon) \cong \mathrm{Tr}_d(F).$$

The equivalence of categories given by the corollary depends on choices of uniformizers for $K$ and $F$.

This corollary allows us to use constructions of extensions of local fields in characteristic $p$ to get extensions in characteristic 0, and vice-versa.

# Making the correspondence explicit

Let $K$ and $F$ be local fields, with $\mathrm{char}(K) = 0$ and $\mathrm{char}(F) = p$. Assume that $K$ and $F$ have the same residue field $k$, and choose uniformizers $\pi_K$, $\pi_F$ for $K$, $F$.

Let $d \le e_K$, and suppose $E/F \in \mathrm{ext}(F)^d$ is totally ramified of degree $n$. Let $\pi_E$ be a uniformizer for $E$ and let

$$g(T) = T^n + \sum_{i=0}^{n-1} a_i T^i$$

be the minimum polynomial for $\pi_E$ over $F$. Write $a_i = a_i(\pi_F)$ as a power series in $\pi_F$ with coefficients in $k$.

Let $A_i(X) \in \mathcal{O}_K[[X]]$ be the series obtained from $a_i(X)$ by replacing each coefficient by its Teichmüller representative in $\mathcal{O}_K$. Set

$$G(T) = T^n + \sum_{i=0}^{n-1} A_i(\pi_K) T^i$$

and let $L$ be generated over $K$ by a root of $G(T)$. Then the equivalence of categories from $\mathrm{ext}(F)^d$ to $\mathrm{ext}(K)^d$ maps $E/F$ to $L/K$.

# Comparison with local class field theory

Let $K$ be a local field of characteristic 0 with finite residue field $k$. Set $F = k((t))$. Suppose $1 \leq d \leq e_K$ and fix a uniformizer $\pi_K$ for $K$. Then there is a group isomorphism

$$F^\times/(1+\mathcal{P}_F^d) \cong K^\times/(1+\mathcal{P}_K^d).$$

To describe this isomorphism let $t^i u(t) \in F^\times$ with $u(t) \in k[[t]]^\times$. Define $U(t) \in \mathcal{O}_K[[t]]$ to be the series obtained by replacing each coefficient in $u(t)$ by its Teichmüller representative in $\mathcal{O}_K$.

Then the isomorphism above maps the coset represented by $t^i u(t)$ onto the coset represented by $\pi_K^i U(\pi_K)$.

By class field theory and the above isomorphism we get a one-to-one correspondence between finite abelian extensions $L/K$ such that $m_{L/K} < d$ and finite abelian extensions $E/F$ such that $m_{E/F} < d$.

This correspondence is in agreement with Deligne's equivalence of categories between $\mathrm{ext}(K)^d$ and $\mathrm{ext}(F)^d$.

# Application to embedding problems

Let $G, \widetilde{G}$ be $p$-groups and let $\pi : \widetilde{G} \to G$ be an onto homomorphism. Let $K$ be a local field with residue characteristic $p$. Let $L/K$ be a finite totally ramified Galois extension and let $\phi : G \to \mathrm{Gal}(L/K)$ be an isomorphism.

The embedding problem associated to this data asks for the following:

- A Galois extension $\widetilde{L}/K$ with $L \subset \widetilde{L}$.
- An isomorphism $\widetilde{\phi} : \widetilde{G} \to \mathrm{Gal}(\widetilde{L}/K)$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\pi} & G \\ {\scriptstyle\widetilde{\phi}}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathrm{Gal}(\widetilde{L}/K) & \longrightarrow & \mathrm{Gal}(L/K) \end{array}$$

If $\mathrm{char}(K) = p$ then this type of embedding problem can always be solved, as shown by Witt [Wi36]. If $\mathrm{char}(K) = 0$ then the embedding problem may or may not have a solution.

# Application to embedding problems (continued)

We wish to use Deligne's result to get sufficient conditions for an embedding problem in characteristic 0 to be solvable.

Let $K$ be a local field of characteristic 0 and let $L/K$ be a totally ramified Galois extension. Let $\pi : \widetilde{G} \to G$ be an onto homomorphism of $p$-groups and let $\phi : G \to \mathrm{Gal}(L/K)$ be an isomorphism.

If $e_K > m_{L/K}$ then there is a $G$-extension $E/F$ of local fields of characteristic $p$ which corresponds to $L/K$. The embedding problem associated to $\pi : \widetilde{G} \to G$ and $E/F$ has a solution $\widetilde{E}/F$.

Can we use Deligne to get a solution $\widetilde{L}/K$ to the original embedding problem in characteristic 0?

It follows from [EK23] that there is $C > 0$, depending only on the group extension $\pi : \widetilde{G} \to G$, such that there exists a solution $\widetilde{E}_0/F$ to the characteristic-$p$ embedding problem satisfying $m_{\widetilde{E}_0/F} \leq Cm_{E/F}$.

Hence if $e_K > Cm_{L/K} = Cm_{E/F}$ then the characteristic-0 embedding problem has a solution $\widetilde{L}_0/K$.

# References

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