Regular subgroups in the holomorph, fixed point free pairs of homomorphisms, and group factorizations

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Holomorph

- Let N be any group. All groups are finite in this talk!!!
- The abstract holomorph of N is the outer semidirect product

$$\mathrm{Hol}(N) = N \rtimes \mathrm{Aut}(N)$$

  with $\mathrm{Aut}(N)$ acting on N naturally.
- The permutational holomorph of N is the inner semidirect product

$$\mathrm{Hol}(N) = \lambda(N) \rtimes \mathrm{Aut}(N) = \rho(N) \rtimes \mathrm{Aut}(N)$$

  as a subgroup of the symmetric group $\mathrm{Sym}(N)$ of N.
- Here $\lambda(N)$ and $\rho(N)$, respectively, denote the subgroups of left and right translations. Explicitly, their elements are given by

  1. $\lambda : N \longrightarrow \mathrm{Sym}(N);\ \ \lambda(\eta) = (x \mapsto \eta x)$
  2. $\rho : N \longrightarrow \mathrm{Sym}(N);\ \ \rho(\eta) = (x \mapsto x\eta^{-1})$

  They are just different ways to identify N as a subgroup of $\mathrm{Sym}(N)$.

Regular subgroups of the holomorph

- A subgroup of $\mathrm{Hol}(N)$ is <u>regular</u> if its action on N is transitive and free.
- Let G be a group of the same order as N.
- The existences of the following are equivalent:
  1. a regular subgroup isomorphic to G inside the holomorph of N
  2. a Hopf-Galois structure of type N on a G-Galois extension of fields
  3. a skew brace with additive group N and circle group G
- **Important Question.** What are some ways to construct regular subgroups isomorphic to G inside the holomorph of N?
- There is a method, due to Byott and Childs, which uses fixed point free pairs of homomorphisms and group factorizations.
- Recall that a pair $f, h : G \longrightarrow \Gamma$ of homomorphisms is said to be <u>fixed point free</u> if $f(\sigma) = h(\sigma)$ only for $\sigma = 1_G$.
- Let me first recall this construction.

Fixed point free pair to regular subgroup

- Let G be a group of the same order as N.
- $\mathrm{Hol}(N) = \lambda(N) \rtimes \mathrm{Aut}(N) = \rho(N) \rtimes \mathrm{Aut}(N)$
- Let $f, h : G \longrightarrow N$ be a fixed point free pair of homomorphisms. Define

$$G_{(f,h)} = \{\lambda(f(\sigma))\rho(h(\sigma)) : \sigma \in G\}.$$

- Notice that we can rewrite

$$\begin{aligned}\lambda(f(\sigma))\rho(h(\sigma)) &= \lambda(f(\sigma))\lambda(h(\sigma)^{-1})\lambda(h(\sigma))\rho(h(\sigma)) \\ &= \lambda(f(\sigma)h(\sigma)^{-1}) \cdot \mathrm{conj}(h(\sigma)).\end{aligned}$$

  Thus h is basically the projection of $G_{(f,h)}$ onto $\mathrm{Aut}(N)$ along $\lambda(N)$.

- Similarly, we can rewrite

$$\begin{aligned}\lambda(f(\sigma))\rho(h(\sigma)) &= \rho(h(\sigma))\rho(f(\sigma)^{-1})\rho(f(\sigma))\lambda(f(\sigma)) \\ &= \rho(h(\sigma)f(\sigma)^{-1}) \cdot \mathrm{conj}(f(\sigma)).\end{aligned}$$

- Thus f is basically the projection of $G_{(f,h)}$ onto $\mathrm{Aut}(N)$ along $\rho(N)$.

Fixed point free pair to regular subgroup

- Let $f, h : G \longrightarrow N$ be a fixed point free pair of homomorphisms. Define

$$\begin{aligned} G_{(f,h)} &= \{\lambda(f(\sigma))\rho(h(\sigma)) : \sigma \in G\} \\ &= \{\rho(h(\sigma)f(\sigma)^{-1}) \cdot \mathrm{conj}(f(\sigma)) : \sigma \in G\}. \end{aligned}$$

  It is a subgroup of $\mathrm{Hol}(N)$ because $\lambda(N)$ and $\rho(N)$ commute element-wise.

- Consider the map $\varphi : G \longrightarrow N$ defined by $\varphi(\sigma) = f(\sigma)h(\sigma)^{-1}$.

- $\varphi(N)$ is then the orbit of 1_N under the action of $G_{(f,h)}$.

$$\begin{aligned} (f,h) \text{ is fixed point free} &\iff \varphi \text{ is injective} \\ &\iff \varphi \text{ is surjective} \\ &\iff G_{(f,h)} \text{ acts transitively on } N \\ &\iff G_{(f,h)} \text{ is a regular subgroup of } \mathrm{Hol}(N) \end{aligned}$$

  It is easy to see that $G_{(f,h)}$ is isomorphic to G in this case.

Fixed point free pair to group factorization

- Let $f, h : G \longrightarrow N$ be a fixed point free pair of homomorphisms.
- Consider the map $\varphi : G \longrightarrow N$ defined by $\varphi(\sigma) = f(\sigma)h(\sigma)^{-1}$.
- That φ is surjective implies that

$$N = f(G)h(G)$$

  and this yields a factorization of the group N.
- **Question.** Suppose that we have factorization

$$N = AB$$

  for some subgroups A and B. Can we construct

  1. a group G of the same order as N
  2. a fixed point free pair of homomorphisms $f, h : G \longrightarrow N$

  such that $A = f(G)$ and $B = h(G)$?

Exact group factorization to fixed point free pair

- **Partial Answer.** Yes under suitable assumptions.
- Suppose that we have an exact factorization

$$N = AB \text{ with } A \cap B = 1$$

for some subgroups A and B. Then, we can construct

1. a group G of the same order as N

$$G = A \times B$$

2. a fixed point free pair of homomorphisms $f, h : G \longrightarrow N$

$$\begin{cases} f(a,b) = a \\ h(a,b) = b \end{cases} \quad \text{for any } a \in A \text{ and } b \in B$$

It is clear that $A = f(G)$ and $B = h(G)$.

Summary

$$\begin{tikzcd}
\boxed{\text{exact group factorization}} \\ N = AB,\ A\cap B = 1 \arrow[dr, "G = A\times B"'] & & \boxed{\text{group factorization}} \\ N = f(G)h(G) \\
& \boxed{\text{fixed point free pair of homomorphisms}} \\ f,h : G \longrightarrow N \arrow[ur] \arrow[d] & \\
& \boxed{\text{regular subgroup in the holomorph of } N} \\ \{\rho(h(\sigma)f(\sigma)^{-1})\cdot \mathrm{conj}(f(\sigma)) : \sigma\in G\} \simeq G &
\end{tikzcd}$$

Generalization

- However, the regular subgroups that can be constructed this way lie inside

$$\lambda(N) \rtimes \mathrm{Inn}(N) = \rho(N) \rtimes \mathrm{Inn}(N).$$

We want to generalize this construction by allowing outer automorphisms.

- We shall restrict to the case when N has trivial center.
- The natural homomorphism conj is then invertible.

$$\mathrm{conj} : N \longrightarrow \mathrm{Inn}(N); \;\; \mathrm{conj}(\eta) = (x \mapsto \eta x \eta^{-1})$$

- The previous construction may be restated as follows.
- Let G be a group of the same order as N.
- Let $f, h : G \longrightarrow \mathrm{Inn}(N)$ be a fixed point free pair of homomorphisms.

$$G_{(f,h)} = \{\rho(\mathrm{conj}^{-1}(h(\sigma)f(\sigma)^{-1})) \cdot f(\sigma) : \sigma \in G\}$$

is a regular subgroup of $\mathrm{Hol}(N)$ which is isomorphic to G.

Fixed point free pair to regular subgroup

- Let N be a centerless group. Let G be a group of the same order as N.
- $\mathrm{Hol}(N) = \lambda(N) \rtimes \mathrm{Aut}(N) = \rho(N) \rtimes \mathrm{Aut}(N)$
- Let $f, h : G \longrightarrow \mathrm{Aut}(N)$ be a fixed point free pair of homomorphisms. Put

$$\begin{aligned} G_{(f,h)} &= \{\rho(\mathrm{conj}^{-1}(h(\sigma)f(\sigma)^{-1})) \cdot f(\sigma) : \sigma \in G\}, \\ &= \{\lambda(\mathrm{conj}^{-1}(f(\sigma)h(\sigma)^{-1})) \cdot h(\sigma) : \sigma \in G\}. \end{aligned}$$

  Here f and h correspond to projections onto $\mathrm{Aut}(N)$ along $\rho(N)$ and $\lambda(N)$.
- However, in order for the above to make sense, we need to assume that

$$f(\sigma) \equiv h(\sigma) \pmod{\mathrm{Inn}(N)} \text{ for all } \sigma \in G.$$

  It is not hard to check that this is a regular subgroup of $\mathrm{Hol}(N)$ which is isomorphic to G.
- In fact, all regular subgroups of $\mathrm{Hol}(N)$ can be constructed this way.

Fixed point free pair to group factorization

- Let $f, h : G \longrightarrow \mathrm{Aut}(N)$ be a fixed point free pair of homomorphisms s.t.

$$(*) \quad f(\sigma) \equiv h(\sigma) \pmod{\mathrm{Inn}(N)} \text{ for all } \sigma \in G.$$

- Let $P = f(G)h(G)$. In general, one can show the following:

  a. $\mathrm{Inn}(N) \leq P \leq \mathrm{Aut}(N)$ and P is a subgroup of $\mathrm{Aut}(N)$
  b. $f(G)\mathrm{Inn}(N) = h(G)\mathrm{Inn}(N) = f(G)h(G)$

  We get a tri-factorization of some subgroup between $\mathrm{Inn}(N)$ and $\mathrm{Aut}(N)$.

- **Question.** Suppose that $\mathrm{Inn}(N) \leq P \leq \mathrm{Aut}(N)$ and we have a factorization

$$P = AB \text{ with } A\mathrm{Inn}(N) = B\mathrm{Inn}(N)$$

  for some subgroups A and B. Can we construct

  1. a group G of the same order as N
  2. a fixed point free pair of homomorphisms $f, h : G \longrightarrow \mathrm{Aut}(N)$ s.t. $(*)$

  such that $A = f(G)$ and $B = h(G)$?

Exact group factorization to fixed point free pair

- **Partial Answer.** Yes under suitable assumptions.
- Suppose that $\mathrm{Inn}(N) \leq P \leq \mathrm{Aut}(N)$ and we have a exact factorization

  $$P = AB,\ A \cap B = 1 \text{ with } A\mathrm{Inn}(N) = B\mathrm{Inn}(N),\ A = (A \cap \mathrm{Inn}(N)) \rtimes C$$

  for some subgroups A and B. Then, we can construct

  1. a group G of the same order as N
     - $G = A \times B$ is too big unless $P = \mathrm{Inn}(N)$
     - $G = (A \cap \mathrm{Inn}(N)) \rtimes_\alpha B$ works because

$$\begin{aligned} |G| &= |A \cap \mathrm{Inn}(N)||B| = \frac{|A \cap \mathrm{Inn}(N)|}{|A|} \cdot |A||B| \\ &= \frac{|\mathrm{Inn}(N)|}{|A\mathrm{Inn}(N)|} \cdot |P| = \frac{|N|}{|P|} \cdot |P| \\ &= |N| \end{aligned}$$

Exact group factorization to fixed point free pair

- Here we let B act on $A \cap \mathrm{Inn}(N)$ via the homomorphism

$$\begin{tikzcd}
B \arrow[r, two heads] \arrow[rrrdd, dashed, bend right, two heads, "\theta"'] & \dfrac{B}{B \cap \mathrm{Inn}(N)} \arrow[d, "\simeq"] & & \\
& \dfrac{B\mathrm{Inn}(N)}{\mathrm{Inn}(N)} \arrow[r, equal] & \dfrac{A\mathrm{Inn}(N)}{\mathrm{Inn}(N)} \arrow[d, "\simeq"] & \\
& & \dfrac{A}{A \cap \mathrm{Inn}(N)} \arrow[r, "\simeq"] & C
\end{tikzcd}$$

and via conjugation by C inside A.

$$\alpha(b)(a) = \theta(b)a\theta(b)^{-1} \text{ for all } a \in A \cap \mathrm{Inn}(N) \text{ and } b \in B$$

Exact group factorization to fixed point free pair

2. a fixed point free pair of homomorphisms $f, h : G \longrightarrow \mathrm{Aut}(N)$ s.t. $(*)$

$$\begin{cases} f(a,b) = a\theta(b) \\ h(a,b) = b \end{cases} \quad \text{for any } a \in A \cap \mathrm{Inn}(N) \text{ and } b \in B$$

It is clear that $A = f(G)$ and $B = h(G)$.

- **Observation.** If $P = \mathrm{Inn}(N)$, then clearly

$$A = A \cap \mathrm{Inn}(N) \text{ and so } C = 1.$$

In particular, the homomorphism $\theta : B \longrightarrow C$ is trivial and

$$G = (A \cap \mathrm{Inn}(N)) \rtimes B = A \times B$$

is simply a direct product, and

$$\begin{cases} f(a,b) = a \\ h(a,b) = b \end{cases} \quad \text{for any } a \in A \text{ and } b \in B$$

are simply the projection maps.

Summary (when N is centerless)

$\boxed{\text{exact group factorization}}$

$$\mathrm{Inn}(N) \leq AB \leq \mathrm{Aut}(N)$$
$$A\mathrm{Inn}(N) = B\mathrm{Inn}(N)$$
$$A \cap B = 1,\ A = (A \cap \mathrm{Inn}(N)) \rtimes C$$

$\boxed{\text{group factorization}}$

$$\mathrm{Inn}(N) \leq f(G)h(G) \leq \mathrm{Aut}(N)$$
$$f(G)\mathrm{Inn}(N) = h(G)\mathrm{Inn}(N)$$

$G = (A \cap \mathrm{Inn}(N)) \rtimes B$

$\boxed{\text{fixed point free pair of homomorphisms}}$

$$f, h : G \longrightarrow \mathrm{Aut}(N) \text{ such that } f(\sigma) \equiv h(\sigma) \pmod{\mathrm{Inn}(N)}$$

$$\updownarrow$$

$\boxed{\text{regular subgroup in the holomorph of } N}$

$$\{\rho(\mathrm{conj}^{-1}(h(\sigma)f(\sigma)^{-1})) \cdot f(\sigma) : \sigma \in G\} \simeq G$$

An application

- **Conjecture.** If N is solvable, then a regular subgroup of $\mathrm{Hol}(N)$ is solvable.
- **Converse.** If N is insolvable, then a regular subgroup of $\mathrm{Hol}(N)$ is insolvable.
- We can characterize the non-abelian simple groups N for which the converse fails to hold, namely the non-abelian simple groups N ...
  1. whose holomorph contains a solvable regular subgroup
  2. which is the type of a Hopf-Galois structure on some solvable extension
  3. which is the additive group of some skew brace with solvable circle group

Theorem (T. 2023, BLMS)

Let N be a non-abelian simple group. The holomorph of N contains a solvable regular subgroup if and only if N is isomorphic to one of the following:

a. $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(8)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$, M_{11};
b. $\mathrm{PSL}_2(q)$ with $q \neq 2, 3$ a prime power.

Forward implication

$\boxed{\text{exact group factorization}}$

$$\mathrm{Inn}(N) \leq AB \leq \mathrm{Aut}(N)$$
$$A\mathrm{Inn}(N) = B\mathrm{Inn}(N)$$
$$A \cap B = 1,\ A = (A \cap \mathrm{Inn}(N)) \rtimes C$$

$\boxed{\text{group factorization}}$

$$\mathrm{Inn}(N) \leq f(G)h(G) \leq \mathrm{Aut}(N)$$
$$f(G)\mathrm{Inn}(N) = h(G)\mathrm{Inn}(N)$$

$G = (A \cap \mathrm{Inn}(N)) \rtimes B$

$\boxed{\text{fixed point free pair of homomorphisms}}$

$$f, h : G \longrightarrow \mathrm{Aut}(N) \text{ such that } f(\sigma) \equiv h(\sigma) \pmod{\mathrm{Inn}(N)}$$

(solvable)

$\boxed{\text{regular subgroup in the holomorph of } N}$

$$\{\rho(\mathrm{conj}^{-1}(h(\sigma)f(\sigma)^{-1})) \cdot f(\sigma) : \sigma \in G\} \simeq G$$

Forward implication

- Suppose that $\mathrm{Hol}(N)$ contains a solvable regular subgroup, G say.
- Then there exist homomorphisms $f, h : G \longrightarrow \mathrm{Aut}(N)$ such that
$$\mathrm{Inn}(N) \leq f(G)h(G) \leq \mathrm{Aut}(N).$$
- Since N is non-abelian simple, the group $f(G)h(G)$ is almost simple whose socle is equal to $\mathrm{Inn}(N) \simeq N$.
- Since G is solvable, the subgroups $f(G)$ and $h(G)$ are also solvable.
- Almost simple groups which are factorizable as the product of two solvable subgroups have been characterized [Li-Xia 2022].
- Their socle must be isomorphic to one of the following:
  - a. $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(8)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$, M_{11};
  - b. $\mathrm{PSL}_2(q)$ with $q \neq 2, 3$ a prime power,

  as stated in the theorem. $\square$

Backward implication

exact group factorization

$$\mathrm{Inn}(N) \leq AB \leq \mathrm{Aut}(N)$$

$$A\mathrm{Inn}(N) = B\mathrm{Inn}(N)$$

$$A \cap B = 1,\ A = (A \cap \mathrm{Inn}(N)) \rtimes C$$

$G = (A \cap \mathrm{Inn}(N)) \rtimes B$

group factorization

$$\mathrm{Inn}(N) \leq f(G)h(G) \leq \mathrm{Aut}(N)$$

$$f(G)\mathrm{Inn}(N) = h(G)\mathrm{Inn}(N)$$

fixed point free pair of homomorphisms

$$f, h : G \longrightarrow \mathrm{Aut}(N) \text{ such that } f(\sigma) \equiv h(\sigma) \pmod{\mathrm{Inn}(N)}$$

(solvable)

regular subgroup in the holomorph of N

$$\{\rho(\mathrm{conj}^{-1}(h(\sigma)f(\sigma)^{-1})) \cdot f(\sigma) : \sigma \in G\} \simeq G$$

Forward implication

- It is enough to show that there exists $\mathrm{Inn}(N) \leq AB \leq \mathrm{Aut}(N)$ for some solvable subgroups A and B satisfying

$$A\mathrm{Inn}(N) = B\mathrm{Inn}(N), \quad A \cap B = 1, \quad A \text{ splits over } A \cap \mathrm{Inn}(N).$$

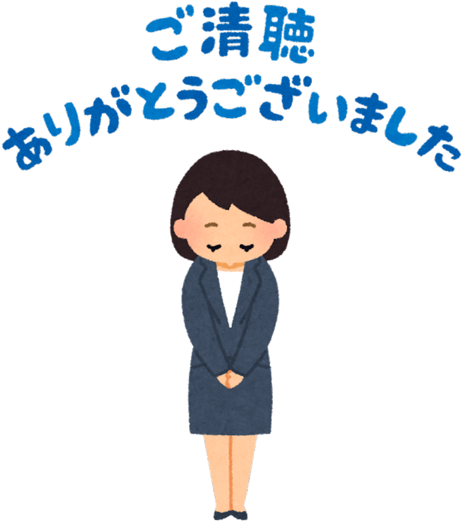
  This is indeed true for all of the N's in question except $N = \mathrm{PSU}_3(8)$.

  a. $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(8)$, $\mathrm{PSU}_4(2)$, M_{11}: MAGMA
  b. $\mathrm{PSL}_2(q)$ with $q \neq 2, 3$ a prime power: Singer cycle and the stabilizer of a one-dimensional subspace

- For the problematic group $N = \mathrm{PSU}_3(8)$, we find a solvable group G of the same order as N and construct a fixed point free pair of homomorphisms $f, h : G \longrightarrow \mathrm{Aut}(N)$ satisfying

$$f(\sigma) \equiv h(\sigma) \pmod{\mathrm{Inn}(N)}$$

  using the help of MAGMA. $\square$



Thank you for listening!