A construction of skew bracoids with a single group

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May 28, 2024

Outline

Bracoids

A *(skew) bracoid* is a triple $(G, N, \odot)$ where G and N are groups and G acts transitively on N via $\odot$ such that

$$g \odot (xy) = (g \odot x)(g \odot e_N)^{-1}(g \odot y)$$

for all $g \in G,\ x, y \in N$.

Bracoids were introduced by Martin-Lyons and Truman to be a generalization of braces, and they correspond to Hopf-Galois structures on (typically non-normal) separable extensions.

2023 Construction

[K.–Truman, 2023] introduce a technique for constructing bracoids using a single group G and an abelian map:

- Let $\psi \in \mathrm{End}(G)$ have abelian image (ψ is an *abelian map*).
- Define $\phi \in \mathrm{Map}(G)$ by $\phi = 1 - \psi$, i.e., $\phi(g) = g\psi(g)^{-1}$ for all $g \in G$.
- Let $H \leq G$ be a subgroup such that $[G, \phi(H)] \leq H$, and let $N = G/H$ be the set of left cosets.
- N has a group structure $\star$ via $xH \star yH = (x\psi(x)^{-1}y\psi(x))H$.
- G acts transitively on N via $g \odot xH = gxH$.
- $(G, N, \odot)$ is a bracoid.

How [KT23 works]. Starting with a group $G = (G,\cdot)$.

- For $\psi \in \mathrm{End}(G)$ abelian we define a binary operation $\circ$ on G via
$$g \circ h = g\psi(g)^{-1}h\psi(g).$$
Then $\mathfrak{B} = (G,\circ,\cdot)$ is a (bi-skew) brace, and $\phi : (G,\circ) \to (G,\cdot)$ is a homomorphism [Koc21].
- For $H \le (G,\cdot)$ we show that H is a strong left ideal of $\mathfrak{B}$ if and only if $[G,\phi(H)] \le G$.
- A construction of [MLT24] then allows for the bracoid structure $(G, G/H, \odot)$.

Example (A special case)

Let $H = \{h \in G : \psi(h) = h\} = \mathrm{Fix}\,\psi = \ker\phi$.
Clearly, $H \le (G,\cdot)$ so $[G,\phi(H)] = \{e_G\} \le H$ and $(G, G/H, \odot)$ is a bracoid.
But $(G/H,\circ) \cong (\phi(G),\cdot) \le G$ via the map $gH \mapsto \phi(g)$.
Using this isomorphism we obtain a transitive action of G on $\phi(G)$:

$$g \odot' \phi(x) = \phi(gx),$$

and hence $(G,\phi(G),\odot')$ is a bracoid.

Objective

Construct bracoids $(G,N,\odot)$ where $N \le G$.

Outline

Following [Koc21], Caranti and Stefanello [CS21] quickly extended the theory beyond abelian maps.

Given $\psi \in \mathrm{End}(G)$ with $\psi([\phi(G), G]) \leq Z(G)$, then $\mathfrak{B} = (G, \circ, \cdot)$ is a biskew brace with

$$g \circ h = g\psi(g)^{-1}h\psi(g).$$

Question. In this more general case, does the condition $[G, \phi(H)] \leq H$ still guarantee that H is a strong left ideal of $\mathfrak{B}$?

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Question. In this more general case, does the condition $[G, \phi(H)] \leq H$ still guarantee that H is a strong left ideal of $\mathfrak{B}$?

Answer. No.

$\psi([\phi(G), G]) \leq Z(G)$

New Question. Let us restrict to the case $H = \operatorname{Fix}\psi$. Does the fact that $[G, \phi(H)] = \{e_G\} \leq H$ still guarantee that H is a strong left ideal of $\mathfrak{B}$?

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Answer. Still, no.

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Answer. Still, no.

But we do have the following.

Proposition

Let $\psi \in \mathrm{End}(G)$, and let $\phi = 1 - \psi$. Then $(G, \phi(G), \odot)$ is a bracoid with $g \odot \phi(x) = \phi(gx)$ if and only if $\psi([\phi(G), G]) = \{e_G\}$.

$\psi([\phi(G), G]) = \{e_G\}$

Example

If $\psi \in \mathrm{End}(G)$ is abelian then $\psi(c) = e_G$ for any commutator c so the condition is satisfied.

Example

If $\psi \in \mathrm{End}(G)$ is idempotent then

$$\psi\phi = \psi(1 - \psi) = \psi - \psi^2 = \psi - \psi = 0$$

so $\phi(g) \in \ker\psi$ for all g and the condition is satisfied.

Example

Let $G = G_1 \times G_2$, $\psi_1 : G_1 \to G_1$ be abelian and $\psi_2 : G_2 \to G_2$ be idempotent, and let $\psi = \psi_1 \times \psi_2$. Then the condition is satisfied.

Outline

Quick refresher

A *set-theoretic solution to the Yang-Baxter equation* is a set B together with a map $r : B \times B \to B \times B$ such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r) : B^3 \to B^3.$$

Write $r(x, y) = (\lambda_x(y), \rho_y(x))$.

We say r is *left non-degenerate* if λ_x is a bijection for all x; otherwise it is *left degenerate*.

Right (non-)degenerate is defined similarly.

It is well-known that skew braces give left and right non-degenerate solutions to the YBE, but in general bracoids do not.

Bracoids to Solutions

In [CKMLT24] we show how, under certain circumstances, bracoids can give left degenerate, right non-degenerate solutions to the YBE, which can be applied here in the case where ψ is idempotent.

For ψ idempotent, let

$$\lambda_x(y) = \psi(x)\phi(y)\psi(x)^{-1}, \ \rho_y(x) = \lambda_x(y)^{-1}xy$$

and $r(x, y) = (\lambda_x(y), \rho_y(x))$.

Notice that

$$\lambda_x(y) = \psi(x)\phi(y)\psi(x)^{-1} = \psi(x)\phi(y)\psi^2(x)^{-1} = \phi(\psi(x)y)$$

hence $\lambda_x(y) \in \phi(G)$ and r is left degenerate.

An example

Let $G = S_n,\ n \geq 2$ and define $\psi \in \mathrm{End}(G)$ by

$$\psi(\sigma) = \begin{cases} \iota & \sigma \in A_n \\ (12) & \sigma \notin A_n \end{cases} \Rightarrow \phi(\sigma) = \begin{cases} \sigma & \sigma \in A_n \\ \sigma(12) & \sigma \notin A_n \end{cases}$$

and $\phi(G) = A_n$. Then

$$r(\sigma, \tau) = \begin{cases} (\tau, \tau^{-1}\sigma\tau) & \sigma \in A_n \\ ((12)\tau(12), (12)\tau^{-1}(12)\sigma\tau) & \sigma \notin A_n,\ \tau \in A_n \\ ((12)\tau, \tau^{-1}(12)\sigma\tau) & \sigma \notin A_n,\ \tau \notin A_n \end{cases}$$

is the solution.

Outline

More examples needed

The condition $\psi([\phi(G), G]) = \{e_G\}$ needs to be better understood.

- Many examples with ψ abelian appear in [Chi13],[Koc21], [Koc22], [KST20], etc.
- Gwen Flaherty (Agnes Scott College) is currently working on the case ψ idempotent.
- Not a lot of “interesting” examples otherwise.

Bracoid Webs?

In [KT24], we adapt the theory of brace blocks to create a *bracoid webs*, a family of bracoids $\{(G_m, N_n, \odot_{m,n}) : m \geq 0,\ n \geq 1\}$ where most, but not all, can be reduced to essentially skew braces.

This construction starts with an abelian map ψ and an H with $[G, \phi(H)] \leq H$.

Question

Can this construction be adapted to any ψ with $\psi([\phi(G), G]) = \{e_G\}$ and $H = \operatorname{Fix}\psi$?

Note that if ψ is idempotent, then $G_m = G_0$ and $N_n = N_1$ for all m, n and the web is not interesting.

A Generalization

Our construction gives the bracoid $(G, \phi(G), \odot)$ with $g \odot \phi(x) = \phi(gx) = g\phi(x)\psi(g)^{-1}$.

Generalizing from $\alpha = \mathrm{id}, \beta = \psi$, we have:

Proposition

Suppose $\alpha, \beta \in \mathrm{End}(G)$ *and* $N = \{\alpha(g)\beta(g)^{-1} : g \in G\} \leq G$. *Then*

$$g \odot x = \alpha(g)x\beta(g)^{-1}, g \in G, \; x \in N$$

gives a bracoid $(G, N, \odot)$.

If we furthermore have $\alpha(g) \neq \beta(g)$ for all $g \neq e_G$ then we have $N = G$ and obtain what Byott and Childs [BC12] call a "fixed-point free pair of homomorphisms" $G \to G$ which can be used to construct HGS on a Galois extension.

References I

Nigel P. Byott and Lindsay N. Childs.
Fixed-point free pairs of homomorphisms and nonabelian Hopf-Galois structures.
New York J. Math., 18:707–731, 2012.

Lindsay N. Childs.
Fixed-point free endomorphisms and Hopf Galois structures.
Proc. Amer. Math. Soc., 141(4):1255–1265, 2013.

Ilaria Colazzo, Alan Koch, Isabel Martin-Lyons, and Paul J. Truman.
Skew bracoids and the Yang-Baxter equation.
arXiv:2404:15929, 2024.

A. Caranti and L. Stefanello.
From endomorphisms to bi-skew braces, regular subgroups, the Yang-Baxter equation, and Hopf-Galois structures.
J. Algebra, 587:462–487, 2021.

Alan Koch.
Abelian maps, bi-skew braces, and opposite pairs of Hopf-Galois structures.
Proc. Amer. Math. Soc. Ser. B, 8:189–203, 2021.

Alan Koch.
Abelian maps, brace blocks, and solutions to the Yang-Baxter equation.
J. Pure Appl. Algebra, 226(9):Paper No. 107047, 2022.

References II

Alan Koch, Laura Stordy, and Paul J. Truman.
Abelian fixed point free endomorphisms and the Yang-Baxter equation.
New York J. Math., 26:1473–1492, 2020.

Alan Koch and Paul J. Truman.
Abelian maps, bracoids, and bracoid webs.
in preparation, 2024.

Isabel Martin-Lyons and Paul J. Truman.
Skew bracoids.
J. Algebra, 638:751–787, 2024.

Thank you.