Skew bracoids and the Yang-Baxter equation

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Overview

Skew bracoids are known to correspond with Hopf-Galois structures on separable, but potentially non-normal, field extensions.

Aim

Show that bracoids can be used to produce and study right nondegenerate solutions of the set-theoretic Yang-Baxter equation.

- Solutions of the set theoretic YBE from skew braces.
- Skew bracoids, and a timeline of their connection with the YBE.
- Skew bracoids containing a skew brace.
- Connections with other algebraic objects.

Set theoretic solutions of the YBE

- A *set theoretic solution of the YBE* on a (nonempty) set G is a map $r : G \times G \to G \times G$ such that

$$(r \times 1)(1 \times r)(r \times 1) = (1 \times r)(r \times 1)(1 \times r)$$

- Henceforth: *a solution on G*.
- Write $r(x, y) = (\lambda_x(y), \rho_y(x))$.
- A solution is called
  - *bijective* if r is a bijection;
  - *left nondegenerate* if each λ_x is bijective;
  - *right nondegenerate* if each ρ_y is bijective;
  - *nondegenerate* if it is both left and right nondegenerate.

Solutions from groups: Lu-Yan-Zhu pairs

Proposition (Lu, Yan, Zhu, 2000)

Let G be a group. Suppose that we have functions $\lambda, \rho : G \to \mathrm{Map}(G)$ such that the following hold for all $x, y \in G$:

- $\lambda_{xy} = \lambda_x \lambda_y$;
- $\rho_{xy} = \rho_y \rho_x$;
- $\lambda_x(y)\rho_y(x) = xy$.

Then

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

is a solution on G.

Skew braces

Definition (Guanieri and Vendramin, 2017)

A *skew brace* is a triple $(G, \star, \cdot)$ where $(G, \cdot)$ and $(G, \star)$ are groups and

$$x \cdot (y \star z) = (x \cdot y) \star x^{-\star} \star (x \cdot z) \text{ for all } x, y, z \in G.$$

- If $(G, \star, \cdot)$ is a skew brace then there is a homomorphism $\gamma : (G, \cdot) \to \mathrm{Aut}(G, \star)$ given by

$$\gamma_x(y) = x^{-\star} \star (x \cdot y),$$

called the γ-*function* of the skew brace.

Solutions from skew braces

Proposition

Let $(G, \star, \cdot)$ be a skew brace. For $x, y \in G$ define

$$\lambda_x(y) = \gamma_x(y) \text{ and } \rho_y(x) = \lambda_x(y)^{-1}xy.$$

Then λ, ρ form a Lu-Yan-Zhu pair on G. The resulting solution $r(x, y)$ is bijective and nondegenerate.

- Each λ_x is bijective, and $\lambda_{xy} = \lambda_x \lambda_y$, by properties of the γ-function.
- Definition of ρ ensures that $\lambda_x(y)\rho_y(x) = xy$.
- "All" that remains in to prove that $\rho_{xy} = \rho_y \rho_x$; bijectivity of each ρ_y follows quickly.
- The inverse solution can be obtained from the *opposite* skew brace.

Skew bracoids

Definition (Martin-Lyons and T, 2024)

A (left) *skew bracoid* is a 5-tuple $(G, \cdot, N, \star, \odot)$ where $(G, \cdot)$ and $(N, \star)$ are groups and $\odot$ is a transitive action of $(G, \cdot)$ on N such that

$$x \odot (\eta \star \mu) = (x \odot \eta) \star (x \odot e_N)^{-1} \star (x \odot \mu)$$

for all $x \in G$ and $\eta, \mu \in N$.

- For brevity: (left) *bracoids*.
- Where possible, write $(G, N, \odot)$, or even (G, N).
- Where possible, write $x \cdot y = xy$ and $\eta \star \mu = \eta\mu$.
- Every skew brace is a bracoid, with $\odot$ and $\cdot$ coinciding.
- If $\mathrm{Stab}_G(e_N) = \{e_G\}$ then (G, N) is *essentially a skew brace*.

A large family of examples

Example

- Let $(G, \star, \cdot)$ be a skew brace and let J be a strong left ideal.
- J is a normal subgroup of $(G, \star)$, so $(G/J, \star)$ is a group.
- J is a subgroup of $(G, \cdot)$, and the cosets of J with respect to $\cdot$ and $\star$ coincide.
- $(G, \cdot)$ acts by left translation on the coset space G/J. Write $\odot$ for this action.
- Then $(G, \cdot, G/J, \star, \odot)$ is a bracoid.

γ-functions of bracoids

- If $(G, N, \odot)$ is a bracoid then there is a homomorphism $\gamma : G \to \mathrm{Aut}(N)$ given by

$$\gamma_x(\eta) = (x \odot e_N)^{-1}(x \odot \eta),$$

  called the *γ-function* of the bracoid.
- In a solution arising from a skew brace we set $\lambda_x(y) = \gamma_x(y)$. This doesn't generalize smoothly to bracoids: the subscript and argument of γ belong to different sets!
- We need some way of "pulling" everything back into G or "pushing" everything onto N.

A short history of bracoids and the YBE

- **April 2023:** Colazzo, Martin-Lyons, T.
  - Let $(G, \star, \cdot)$ be a skew brace, let J be a strong left ideal, and consider the bracoid $(G, \cdot, G/J, \star, \odot)$.
  - Suppose that there exists $H \subseteq G$ that is a complement for J in both $(G, \star)$ and $(G, \cdot)$.
  - Define $\lambda_x(y) = \gamma_x(yJ) \cap H$ and $\rho_y(x) = \lambda_x(y)^{-1}xy$.
  - Then λ, ρ form a Lu-Yan-Zhu pair on G, giving a right-nondegenerate solution.
- **August 2023:** Koch, T.
  - Let $G = (G, \cdot)$ be a group, H a subgroup of G, and consider a bracoid of the form $(G, \cdot, H, \cdot, \odot)$.
  - H acts on itself via $\odot$. Assume that $h \odot e = h$ for all $h \in H$.
  - Define $\lambda_x(y) = \gamma_x(y \odot e)$ and $\rho_y(x) = \lambda_x(y)^{-1}xy$.
  - Then λ, ρ form a Lu-Yan-Zhu pair on G, giving a right-nondegenerate solution.

The common factor

Let (G, N) be a bracoid and let $S = \mathrm{Stab}_G(e_N)$.

Suppose that S has a complement H in G, so that G has an exact factorization $G = HS$. Then:

- (H, N) is a bracoid;
- $\mathrm{Stab}_H(e_N) = \{e_G\}$.

Hence (H, N) is essentially a skew brace.

We shall say that (G, N) *contains a skew brace* (H, N).

There is a bijection $b : N \to H$ defined by $b(\eta) \odot e_N = \eta$.

Solutions from bracoids containing a skew brace

Theorem

Suppose that (G, N) is a bracoid containing a skew brace (H, N). For $x, y \in G$ define

$$\lambda_x(y) = b(\gamma_x(y \odot e_N)) \in H$$

and

$$\rho_y(x) = \lambda_x(y)^{-1}xy.$$

Then λ and ρ form a Lu-Yan-Zhu pair on G, and

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

is a right nondegenerate solution on G.

Examples

Example

If (G, N) is essentially a skew brace then our construction yields the expected solution.

Example

If $(G, \star, \cdot)$ is a skew brace and J is a strong left ideal then in the bracoid $(G, \cdot, G/J, \star, \odot)$ we have $\mathrm{Stab}_G(eJ) = J$. If this has a complement in G then our construction yields the same solution as in the CMLT approach.

Example

If we have a bracoid of the form $(G, \cdot, H, \cdot, \odot)$ with $h \odot e = h$ for all $h \in H$ then H is a complement to $S = \mathrm{Stab}_G(e)$; our construction yields the same solution as the KT approach.

Examples

Example (Byott, 2024)

- Let N be an elementary abelian group of order 8.
- Then $\mathrm{Hol}(N)$ contains a transitive subgroup $G \cong \mathrm{GL}_3(\mathbb{F}_2)$.
- We may form the bracoid $(G, N, \odot)$, where $\odot$ is the natural action of G on N.
- We have $|G| = 168$, so $|S| = 21$, and so G has a exact factorization $G = HS$ with H a Sylow 2-subgroup of G.
- Hence our construction applies.

Do all skew bracoids contain a skew brace?

Example (Darlington, 2024)

- Let p and q be prime numbers with $p \equiv 1 \pmod{q^2}$, and let N be a cyclic group of order pq.
- Then $\mathrm{Hol}(N)$ contains a minimally transitive subgroup G of order pq^2.
- We may form the bracoid $(G, N, \odot)$, where $\odot$ is the natural action of G on N.
- The subgroup $S = \mathrm{Stab}(e_N)$ does not have a complement in G.

Subsolutions

Recall: (G, N) is a bracoid containing a skew brace (H, N). We have defined

$$\lambda_x(y) = b(\gamma_x(y \odot e_N)) \in H \text{ and } \rho_y(x) = \lambda_x(y)^{-1}xy.$$

Proposition

The solution $r(x, y) = (\lambda_x(y), \rho_y(x))$ restricts to each of H and S.

Proof.

We have $\lambda_x(y) \in H$ by construction.

If in addition $x, y \in H$ then $\rho_y(x) \in H$.

If $x, y \in S$ then $\lambda_x(y) = b((x \odot e_N)^{-1}(xy \odot e_N)) = e$ and $\rho_y(x) = xy$. □

Recovering the whole solution

Theorem (Catino, Colazzo, Stefanelli, 2020)

Given sets X, Y, solutions r_X, r_Y on these sets, and maps $\alpha : Y \to \mathrm{Perm}(X)$ and $\beta : X \to \mathrm{Perm}(Y)$, all satisfying various compatibility conditions, we can construct a solution

$$r_X \bowtie r_Y : (X \times Y) \times (X \times Y) \to (X \times Y) \times (X \times Y),$$

called the matched product of the solutions r_X and r_Y (via α and β).

Proposition

(G, N) is a bracoid containing a skew brace (H, N) then the right nongenerate solution we obtain is isomorphic to the matched product of a solution on $X = H$ and a solution on $Y = S$.

Connections with other algebraic objects

Definition

A left *semibrace* is a triple $(G, +, \cdot)$ in which $(G, \cdot)$ is a group, $(G, +)$ is a left cancellative semigroup, and we have

$$x \cdot (y + z) = x \cdot y + x \cdot (x^{-1} + y) \text{ for all } x, y, z \in G.$$

We have $(G, +) = (G + e, +) \oplus (E, +)$, where $(G + e, +)$ is a group and E is the set of idempotents with respect to $+$.

Theorem (Catino, Colazzo, Stefanelli, 2017)

Let $(G, +, \cdot)$ be a left semibrace and for $x, y \in G$ define

$$\mathcal{L}_x(y) = x(x^{-1} + y) \text{ and } \mathcal{R}_y(x) = \mathcal{L}_x(y)^{-1}xy.$$

Then $\mathcal{L}, \mathcal{R}$ form a Lu-Yan-Zhu pair on G and $s(x, y) = (\mathcal{L}_x(y), \mathcal{R}_y(x))$ is a left nondegenerate solution on G.

Finding the right sort of connection

Recall:

- **left** bracoids containing a skew brace yield **right** nondegenerate solutions
- **left** semibraces yield **left** nondegenerate solutions.

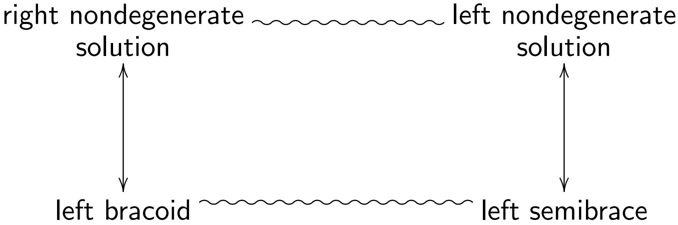
We could establish a connection of the following form:

$$\text{left bracoid} \quad \leftrightsquigarrow \quad \text{right nondegenerate solution} \quad \leftrightsquigarrow \quad \text{right semibrace}$$

In the extreme case, this would connect a left skew brace with a right skew brace: undesirable.

Finding the right sort of connection

Instead, we establish a connection of the following form:



In the extreme case both objects reduce to left skew braces.

If (G, N) is a bracoid containing a skew brace (H, N) then we may transport the structure of N to H, so **without loss of generality we work with bracoids of the form $(G, \cdot, H, \star, \odot)$.**

The right sort of connection

Theorem

Let $G = (G, \cdot)$ be a group and let H, S be subgroups of G. There is a bijection between

1. *binary operations $\star$ on H and transitive actions $\odot$ of G on H such that $(G, \cdot, H, \star, \odot)$ is a left bracoid containing a skew brace $(H, \cdot, \star)$ and with $\mathrm{Stab}_G(e) = S$;*
2. *binary operations $+$ on G such that $(G, +, \cdot)$ is a left semibrace in which $G + e = H$ and $E = S$.*

- Given a suitable bracoid $(G, \cdot, H, \star, \odot)$ let $\lambda_x(y) = \gamma_x(y \odot e)$ for $x, y \in G$ and define $x + y = y\lambda_{y^{-1}}(x)$.
- Given a suitable left semibrace $(G, +, \cdot)$ define $h \star k = k + h$ for $h, k \in H$ and $x \odot h = xh + e$ for $x \in G$ and $h \in H$.

What about solutions?

Proposition

Suppose that the left bracoid $(G, \cdot, H, \star, \odot)$ and the left semibrace $(G, +, \cdot)$ correspond as in the Theorem on the previous slide.
Let r be the right nondegenerate solution arising from $(G, \cdot, H, \star, \odot)$, and let s be the left nondegenerate solution arising from $(G, +, \cdot)$.
Then

$$s(x, y) = \mu r \mu^{-1}(x, y),$$

where $\mu(x, y) = (y^{-1}, x^{-1})$.

Some natural questions...

- What happens if S has a **normal** complement in G? (We can answer this one.)
- What is the effect of varying the complement H to S in G? Different complements need not be isomorphic...
- What do methods for constructing semibraces tell us about bracoids, or vice versa?
- $\ldots$?

Thank you for your attention.