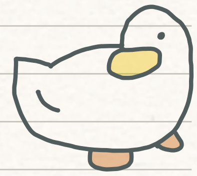
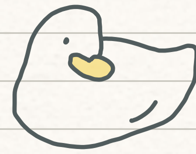
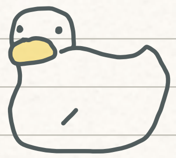


Hopf-Galois structures of cyclic type on parallel extensions of prime power degree

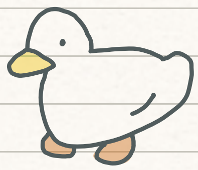


Joint work with Cindy Tsang

- $\rightsquigarrow$ HGT recap
- $\rightsquigarrow$ parallel extensions
- $\rightsquigarrow$ $[L:K] = p^e$, HGS of cyclic type



Quick recap



L, K fields, H a K-Hopf algebra

If

↳ L is a H-module algebra

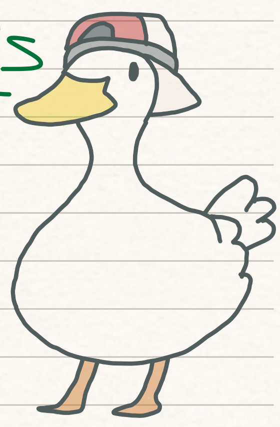
↳ The K-linear map

$$j : L \otimes_K H \longrightarrow \mathrm{End}_K(L)$$

$$x \otimes h \longmapsto j_{x\otimes h}(y) = x(h \cdot y)$$

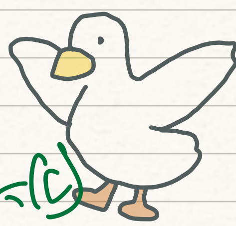
is bijective

Then $(H, \cdot)$ gives a Hopf-Galois structure on L/K.

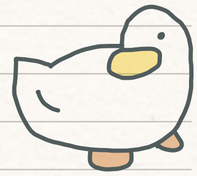


e.g.

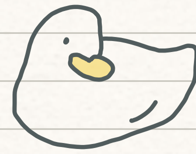
L/K finite, $G \leq \mathrm{Aut}_K(L)$



$K[G]$-Galois $\Longleftrightarrow$ Galois with group G

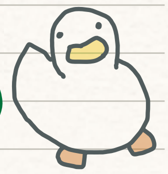


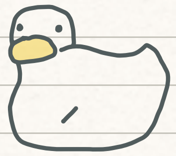
Greither–Pareigis



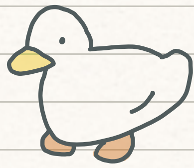
$$\begin{tikzcd} E \arrow[d, dash] \arrow[d, bend left=60, dash, "G'"] \arrow[dd, bend right=60, dash, "G"'] \\ L \arrow[d, dash] \\ K \end{tikzcd}$$

$\lambda : G \longrightarrow \mathrm{Perm}(G/G')$





HGSs on L/K $\longleftrightarrow$ $N \leq \mathrm{Perm}(G/G')$ regular & normalised by $\lambda(G)$



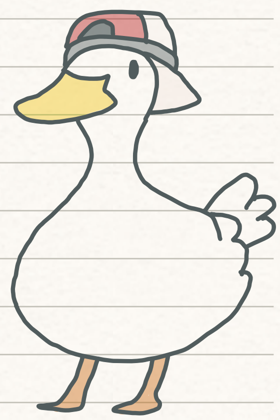
HGS of <u>type</u> $[N]$ $\longleftrightarrow$ N

Byott's translation

HGSs of type N on L/K $\longleftrightarrow$ $M \overset{\text{trans.}}{\leq} \mathrm{Hol}(N) \cong N \rtimes \mathrm{Aut}(N)$ $(< \mathrm{Perm}(N))$

$G \overset{\phi}{\cong} M$

$\phi(G') = \mathrm{Stab}_M(1_N)$



Parallel extensions

$$\begin{tikzcd}
E \arrow[d, dash] \arrow[dr, dash] & \\
L \arrow[d, dash, "n"'] & L' \arrow[dl, dash, "n"] \\
k &
\end{tikzcd}$$

L'/k is parallel to L/k

The problem

HGT of L/k $\overset{?}{\longleftrightarrow}$ HGT of L'/k

Q1

L/k admits an HGS $\overset{?}{\longleftrightarrow}$ L'/k admits an HGS

Examples

(1)

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\sqrt[4]{2}, i) \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{Q}(\sqrt[4]{2}) \arrow[d, no head, "4"] & & \mathbb{Q}(\sqrt{2}, i) \arrow[dll, no head, "4"] \\
\mathbb{Q} & &
\end{tikzcd}$$

non-Galois

Galois

	C_4	$C_2 \times C_2$
$\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$	1	1
$\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$	3	1

(2) L/K, L'/K conjugate

L/K has a HGS of type N

$\iff$

L'/K has a HGS of type N

([L:K] = Burnside is a special case)

$\rightsquigarrow [L:K] = pq$

In general, Q1 is very hard...

$\rightsquigarrow$ restrict the type?

Q2

$$\begin{array}{ccc} L/K \text{ HGS} & \overset{?}{\longleftrightarrow} & L'/K \text{ HGS} \\ \text{type } N & & \text{type } N \end{array}$$

Idea

$H \leq G$, $(G:H) = n$, $C = \mathrm{core}_G(H)$

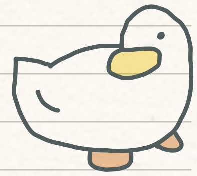
$$\begin{tikzcd}
E \arrow[r, dash] \arrow[d, dash] \arrow[d, bend left, dash, "G'"] \arrow[dd, bend right, dash, "G"'] & E^C \arrow[d, dash] \arrow[d, bend left, dash, "H/C"] \arrow[dd, bend right, dash, "G/C"'] \\
L \arrow[d, dash, "n"] & E^H \arrow[d, dash, "n"] \\
K & K
\end{tikzcd}$$

Q2a

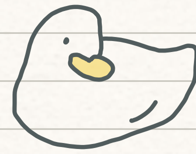
Given $(G, G') \cong M \leq \mathrm{Hol}(N)$

$\exists M^* \leq \mathrm{Hol}(N)$, $(G/C, H/C) \cong M^*$?

There is typically no natural way to see M/C as a subgroup of $\mathrm{Hol}(N)$!



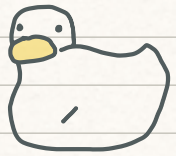
A note on bracoids



$(G, N, \odot)$ bracoid, $\mathrm{Stab}_\odot(1_N) = G'$

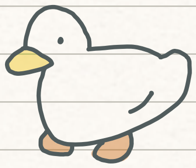
"L/K HGS type N" $\Longrightarrow$ "L'/K HGS type N"

(G, G') $\qquad$ $(G/C, H/C)$



$\rightsquigarrow \exists \odot'$, $(G/C, N, \odot')$ bracoid

s.t. $\mathrm{Stab}_{\odot'}(1_N) = H/C$



cyclic prime power order

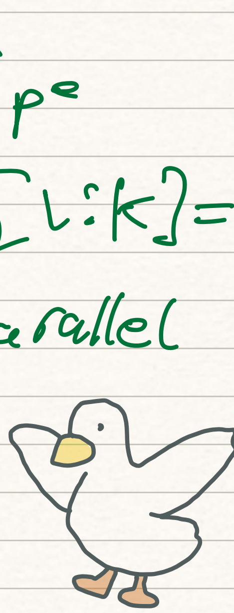
Rest of talk:

↳ p prime $\quad$ ↳ $N = C_{p^e}$

↳ $e \in \mathbb{Z}_{>0}$ $\quad \longrightarrow [L:K] = p^e$

L/K (non-norm., sep), L'/K parallel

Suppose L/K admits HGS of type N



Aim

Answer Q2

Let $N = \langle \sigma \rangle$, $\varphi_b \in \mathrm{Aut}(N)$

$$\varphi_b(\sigma) = \sigma^b$$

Main results (D.-Tsang)

Theorem 1

Let $p > 2$.

L'/K admits HGS type N

$$\Longleftrightarrow$$

L/K & L'/K conjugate

Theorem 2

Let $p=2$. L'/K does not admit HGS type N iff one of.

(i) $|H\cap N| \geq 4$

(ii) $|H\cap N| = 2$ & $\nexists x\in G$, $\mathrm{ord}(x) = 2^e$

(iii) $|H\cap N| = 2$ & $H \ntrianglelefteq G$

(iv) $|H\cap N| = 1$, $|G| = 2^{e+1}$,
$H = \langle [\sigma^u, \varphi_{-1}] \rangle$, odd u
$G' = \langle \varphi_{1+2^{e-1}} \rangle$,

$H\cap N \leq C$

$$\begin{cases} |G\cap N| \geq 8, \text{ or} \\ |G\cap N| = |[G,G]| = 4 \end{cases}$$

$G/C \cong G$

H

Theorem 1 (Sketch proof)

$(\Leftarrow)$ trivial

$(\Rightarrow)$ $G \leq \mathrm{Hol}(N) \longleftrightarrow L/K$

$|H| = p^s$

WTS $\exists g \in G,\ gHg^{-1} = \mathrm{Stab}_G(N) = G'$

<u>Lemma (D.–Tsang)</u>

(a) $T \leq \mathrm{Hol}(N)$ trans

$\Rightarrow \exists x \in T,\ \mathrm{ord}(x) = p^e$

(b) $|H \cap N| > 1 \Rightarrow \nexists x \in G/C,\ \mathrm{ord}(x) = p^e$

$\rightsquigarrow$ can assume $|H \cap N| = 1$

<u>Lemma (Crespo)</u>

$Q = \mathrm{Syl}_p(\mathrm{Hol}(N))$

$\rightsquigarrow$ G trans. $\Longleftrightarrow$ $G \cap Q$ trans.

$\rightsquigarrow$ Can assume G is a p-group

$\mathrm{im}\left(H \xrightarrow{N} \mathrm{Aut}(N)\right) \cong H$

$\rightsquigarrow |H \cap N| = 1$

In particular, $\mathrm{Aut}(N)$ is cyclic

$\Rightarrow H$ is cyclic

$$\rightsquigarrow \quad H = \langle [\sigma^{u}, \varphi_a] \rangle$$

$$\left.\begin{array}{l} a \equiv 1 + p^{e-s} \mod p^e \\ u \equiv 0 \mod p^{e-s} \end{array}\right\}$$

$$\Rightarrow \exists v, \; v(1-a) \equiv -u \pmod{p^e}$$

G trans $\Rightarrow \exists g = [\sigma^v, \varphi_b] \in G$

$$\rightsquigarrow g[\sigma^u, \varphi_a]g^{-1} = [\sigma^{u(b-1)}, \varphi_a]$$

$\varphi_a \in Aut(N)$, $a \equiv 1 \mod p$

observation:

$$v_p(u(b-1)) = v_p(u) + \underbrace{v_p(b-1)}_{\geq 1} > v_p(u)$$

so repeat!

$$H \sim \langle \varphi_a \rangle \leqq G' \quad \square$$