Classifying bidihedral skew braces part 2: the classification

Alan Koch

Agnes Scott College

May 27, 2026

Joint work with Paul J. Truman

Some notation is required

We will be exploring many operations on the same underlying set.

Let G be a group with binary operation $\diamond$.

Write

- $G^\diamond = (G, \diamond)$
- $Z^\diamond = Z(G, \diamond)$
- $g^{\diamond n} = \underbrace{g \diamond g \diamond \cdots \diamond g}_{n \text{ times}}$
- For $S \subseteq G$, $\langle S \rangle_\diamond$ subgroup of $G^\diamond$ generated by S.
- $e \in G$ the identity

If the operation is $g \bullet h$ (or gh) we write g^n for $g^{\bullet n}$ and write $G^\bullet, Z^\bullet, \langle S \rangle_\bullet$.

Notation for inverses will be handled on a case-by-case basis.

A (skew left) brace (hereafter, brace) is a triple $(G, \bullet, \circ)$ where

- $G^\bullet$ is a group ($\bullet$ is the additive operation)
- G° is a group ($\circ$ is the multiplicative operation)
- the following brace relation holds for all $g, h, \ell \in G$:

$$g \circ (h\ell) = (g \circ h)g^{-1}(g \circ \ell), \; gg^{-1} = e.$$

Easy example: let $\circ = \bullet$. $(G, \bullet, \bullet)$ is the trivial brace.

Fact. In a brace, both $G^\bullet$ and G° have the same identity element.

If both $(G, \bullet, \circ)$ and $(G, \circ, \bullet)$ are braces, then we have a biskew brace.

Objective

Obtain a complete set of (isomorphism classes of) braces $(G, \bullet, \circ)$ with

$$G^\bullet \cong G^\circ \cong D_n,$$

D_n the dihedral group of order $2n$, $n \geq 3$.

Such braces are called *bidihedral*.

Strategy. Fix $G^\bullet$ and construct all possible G°.

Throughout, we assume

$$G^\bullet = \langle r, s : r^n = s^2 = rsrs = e \rangle = \{r^i, r^i s : 0 \leq i \leq n-1\},$$

and we let

$$K = K_n = \{k \in \mathbb{Z}_n : k^2 = 1\}, \ \kappa = \kappa_n = |K|.$$

The classification will consist of three families of braces.

Outline

The family $\mathscr{A}_k$

The family $\mathscr{B}_k$

The family $\mathscr{C}_k$

The full classification

Bidihedral braces and Hopf-Galois structures

What's next

$G^{\bullet} = \langle r, s : r^n = s^2 = rsrs = e \rangle$

Fix $k \in K$.

Define

$$r^i s^u \circ r^j s^v = \left(r^{k^v i} s^u\right)\left(r^{k^u j} s^v\right) = r^{k^v i + (-k)^u j} s^{u+v},$$

i.e.,

$$\begin{aligned} r^i \circ r^j &= (r^i)(r^j) = r^{i+j} \\ r^i \circ r^j s &= (r^{ki})(r^j s) = r^{ki+j} s \\ r^i s \circ r^j &= (r^i s)(r^{kj}) = r^{i-kj} s \\ r^i s \circ r^j s &= (r^{ki} s)(r^{kj} s) = r^{k(i-j)} s. \end{aligned}$$

$r^i s^u \circ r^j s^v = r^{k^v i + (-k)^u j} s^{u+v}$

G° is a group:

- $g \circ e = e \circ g = g$ (where $e \in G^\bullet$ is the identity)
- $r^i \circ r^{-i} = e = r^i s \circ r^i s$ (inverses)
- G° is associative (straightforward)
- $G^\circ = \langle r, s : r^{\circ n} = s \circ s = r \circ s \circ r \circ s = e \rangle$

so $G^\circ \cong D_n$.

Not hard to show $(G, \bullet, \circ)$ satisfies the brace relation.

So, $\mathscr{A}_k := (G, \bullet, \circ)$ is a brace.

Although

$$G^\bullet = \langle r, s : r^n = s^2 = rsrs = e \rangle$$
$$G^\circ = \langle r, s : r^{\circ n} = s \circ s = r \circ s \circ r \circ s = e \rangle,$$

if $k \neq 1$ then $\mathscr{A}_k$ is not the trivial brace.

For example,

$$r \circ s = r^k s,\ \ r^k \circ s = rs,$$

so the map $r \mapsto r,\ s \mapsto s$ is not an isomorphism $\mathscr{A}_k \to (G, \bullet, \circ)$.

We get a family of braces, depending on $k \in K$.

Recall $\gamma_g[h] = g^{-1}(g \circ h)$, and note

$$\begin{aligned}
\gamma_r[r] &= r^{-1}(r \circ r) = r \\
\gamma_r[s] &= r^{-1}(r \circ s) = r^{-1} r^k s = r^{k-1} s \\
\gamma_s[r] &= s(s \circ r) = s r^{-k} s = r^k \\
\gamma_s[s] &= s(s \circ s) = s.
\end{aligned}$$

These are the braces predicted by Paul in part 1 ($d = n$).

From Part I

Recall $d = n$ (possibly $d = n/2$ if $8 \mid n$).

$$\gamma_r[r] = r^{1+d} \qquad \gamma_r[s] = r^{(1+d)k-1} s$$
$$\gamma_s[r] = r^k \qquad \gamma_s[s] = s.$$

Question. When is $\mathscr{A}_{k'} \cong \mathscr{A}_k$?

General strategy:

- Pick $\alpha \in \mathrm{Aut}(G^\bullet)$.
- Define an operation $\circ'$ on G by

$$g \circ' h = \alpha^{-1}(\alpha(g) \circ \alpha(h)).$$

- Then $(G, \bullet, \circ')$ is a brace, isomorphic to $\mathscr{A}_k$.
- Turns out $(G, \bullet, \circ') = \mathscr{A}_{k'}$ for some $k' \in K$.

Fact. Every automorphism of $G^\bullet$ preserves $\circ$.

Thus, $\mathscr{A}_{k'} \cong \mathscr{A}_k$ if and only if $k' = k$.

Every choice of $k \in K$ gives a different bidihedral brace.

$r^i s^u \circ r^j s^v = r^{k^v i + (-k)^u j} s^{u+v}, \; \mathscr{A}_k = (G, \bullet, \circ)$

Some properties:

- If $k = 1$ we get the trivial brace $(G, \bullet, \bullet)$.
- If $\widehat{\mathscr{A}_k}$ is the opposite brace to $\mathscr{A}_k$, then $\widehat{\mathscr{A}_k} = \mathscr{A}_{-k}$.
- Each $\mathscr{A}_k$ is biskew, and $(G, \circ, \bullet) \cong \mathscr{A}_k$.
- $H := \langle r \rangle_\bullet = \langle r \rangle_\circ$ is an ideal of $\mathscr{A}_k$ which is a trivial subbrace of $\mathscr{A}_k$.
- If $2 \nmid n$ this is a complete list of bidihedral braces.
- Each $\mathscr{A}_k$ is also skew right, that is,

$$(xy) \circ a = (x \circ a) a^{-1} (y \circ a), \; x, y, a \in G.$$

Outline

Alan Koch

Abelian maps: a review

One simple way to construct braces is using abelian maps.

Let $\Gamma := \Gamma^\bullet$ be any group, and let $\psi \in \text{End}(\Gamma)$ with $\psi(\Gamma) \leq \Gamma$ abelian.

Then ψ is said to be an *abelian map*, and we denote the set of all abelian maps on Γ by $\text{Ab}(\Gamma)$.

For $\psi \in \text{Ab}(\Gamma)$ we define:

$$x * y = x\psi(x)^{-1}y\psi(x), \; x, y \in \Gamma.$$

Then $(\Gamma, \bullet, *)$ is a (biskew) brace (K, 2021).

This theory has been extended several ways through the works of Caranti-Stefanello, Stefanello-Trappeniers (early 2020's).

Fact. If Γ is finite and ψ has no nontrivial fixed points (*fixed-point-free*), then $\Gamma^* \cong \Gamma^\bullet$.

One isomorphism $\Gamma^* \to \Gamma^\bullet$ is $x \mapsto x\psi(x^{-1})$.

The dihedral case

Fixed-point-free abelian maps on dihedral groups have been classified by (Childs, 2013) and (K-Stordy-Truman, 2020).

If n is odd, then the only such map is the trivial map, resulting in the trivial brace.

If n is even we also have a family of maps $\psi = \psi_c,\ 0 \leq c < n$ given by

$$\psi(r) = r^c s,\ \psi(s) = (r^c s)^c.$$

If $0 \leq c, c' \leq n$ give groups $G^*, G^{*'}$ respectively then $g *' h = g * h$ for all $g, h \in G$ iff $4 \mid n$ and $c' = c + n/2$.

We assume for the rest of this section that n is even.

Idea.

- Fix $k \in K$, giving $\mathscr{A}_k = (G, \bullet, \circ)$.
- The abelian maps on G° are determined by their values on r and s, namely

$$\psi_c(r) = r^c s,\ \psi_c(s) = (r^c s)^c.$$

- Note that $r^c s \circ r^c s = e$.
- Let $\psi = \psi_c \in \mathrm{Ab}(G^\circ)$, and define

$$g * h = g \circ \psi(g) \circ h \circ \psi(g).$$

- Turns out $g * h = g \circ (\psi(g) h \psi(g))$.
- Abelian map theory gives that $(G, \circ, *)$ is a brace.

Question. Is $(G, \bullet, *)$ a brace?

$g * h = g \circ (\psi(g)h\psi(g))$. Is $(G, \bullet, *)$ brace?

Yes. Let $a = \psi(g)$. Notice $a^2 = a^{\circ 2} = a^{*2} = e$.

$$\begin{aligned} g * (h\ell) &= g \circ (a(h\ell)a) \\ &= (g \circ a)g^{-1}(g \circ h)g^{-1}(g \circ \ell)g^{-1}(g \circ a) \\ (g * h)g^{-1}(g * \ell) &= (g \circ (aha))g^{-1}(g \circ (a\ell a)) \\ &= (g \circ a)g^{-1}(g \circ h)g^{-1}(g \circ a) \\ &\quad\; g^{-1}(g \circ a)g^{-1}(g \circ \ell)g^{-1}(g \circ a) \end{aligned}$$

So the brace relation holds if and only if

$$(g \circ a)g^{-1}(g \circ a)g^{-1} = e$$

Fact. In a skew left brace, $x^{-1}(x \circ y)x^{-1} = (x \circ y^{-1})^{-1}$.
(True since $x = x \circ (yy^{-1}) = (x \circ y)x^{-1}(x \circ y^{-1})$.)
So we need $(g \circ a)(g \circ a)^{-1} = e$ which is true.

$g * h = g \circ (\psi(g) h \psi(g)),\ \psi$ nontrivial.

For now, the brace $(G, \bullet, *)$ will be denoted $\mathscr{B}_{k,c}$.

Recall $r^i s^u \circ r^j s^v = \left(r^{k^v i} s^u\right)\left(r^{k^u j} s^v\right) = r^{k^v i + (-k)^u j} s^{u+v}$.

Since

$$r \circ r = r^2,\ r * r = r \circ (r^c s) r (r^c s) = r \circ r^{-1} = e$$

we see that this is a "new" brace.

Also, since $(G, \bullet, \circ)$ and $(G, \bullet, *)$ have the same additive group $G^\bullet$ and $\mathrm{Aut}(G, \bullet, \circ) = \mathrm{Aut}(G^\bullet)$ it follows that $\mathscr{B}_{k,c} \ncong \mathscr{A}_{k'}$ for every $k, k' \in K,\ 0 \le c < n$.

Question. When is $\mathscr{B}_{k',c'} \cong \mathscr{B}_{k,c}$?
Let $\alpha \in \mathrm{Aut}(G^\bullet)$ be given by $\alpha(r) = r,\ \alpha(s) = rs$.
Let $\mathscr{B}_{k,c} = (G, \bullet, *)$ and define

$$g *' h = \alpha(\alpha^{-1}(g) * \alpha^{-1}(h)).$$

Notice, for example, that

$$r * s = r \circ (r^c s \bullet s \bullet r^c s) = r \circ r^{2c} s = r^{k+2c} s$$

and

$$\begin{aligned} r *' s &= \alpha(r * r^{-1} s) \\ &= \alpha(r \circ (r^c s \bullet r^{-1} s \bullet r^c s)) \\ &= \alpha(r \circ r^{2c+1} s) \\ &= \alpha(r^{k+2c+1} s) \\ &= r^{k+2(c+1)s}. \end{aligned}$$

More generally, turns out $\mathscr{B}_{k,c+1} \cong \mathscr{B}_{k,c}$ for all c.

So the brace obtained (up to isomorphism) is independent of the choice of c.

Let $\psi(r) = s,\ \psi(s) = e$ $(c = 0)$, and let $\mathscr{B}_k = \mathscr{B}_{k,0}$.

The operation is

$$r^i s^u * r^j s^v = r^i s^u \circ r^{(-1)^i j} s^v = r^{k^v i + (-1)^{i+u} k^u j} s^{u+v}.$$

Let $\beta_u(r) = r^u, \beta(s) = s$, where $\gcd(u, n) = 1$. Then $\beta_u \in \mathrm{Aut}(G^\bullet)$. If

$$g *' h = \beta_u(\beta_u^{-1}(g) * \beta_u^{-1}(h)).$$

Then $(G, \bullet, *') = \mathscr{B}_{k,cu} \cong \mathscr{B}_{k,c}$.

As $\mathrm{Aut}(G^\bullet) = \langle \alpha, \beta_u : \gcd(u, n) = 1 \rangle$ we get $\mathscr{B}_{k'} \cong \mathscr{B}_k$ if and only if $k' = k$.

So all the $\mathscr{B}_k$ are distinct.

$g * h = g \circ (\psi(g)h\psi(g)),\ \psi(r) = s,\ \psi(s) = e$

When we look at the γ-functions we get, for example,

$$\gamma_{r^k s}[r^k s] = r^{k-2} s \qquad \gamma_{r^k s}[r] = r^{-k}$$
$$\gamma_r[r^k s] = r^{-k-1} s \qquad \gamma_r[r] = r^{-1}$$

which agrees with part 1:

From Part 1

For $G^\circ = \langle a, b \rangle$:

$$\gamma_a(a) = b^{-2} a \qquad \gamma_a(b) = b^{k'}$$
$$\gamma_b(a) = b^{k'-1} a \qquad \gamma_b(b) = b^{-1}$$

Let $a = r^k s,\ b = r,\ k' = -k$.

$g * h = g \circ (\psi(g)h\psi(g))$

More properties of $\mathscr{B}_k$:

- If $\gamma_{\mathscr{A}_k}$ and $\gamma_{\mathscr{B}_k}$ are the γ-functions on these braces, then

$$\gamma_{\mathscr{B}_k,g}[h] = \gamma_{\mathscr{A}_k,g}[\psi(g)h\psi(g)].$$

- For $k \neq 1$, $\mathscr{B}_k$ does not arise from a $\psi \in \mathrm{Ab}(G^\bullet)$ using the usual abelian map construction.
- $\mathscr{B}_k$ is biskew iff $k = 1$.
- $\widehat{\mathscr{B}_k} = \mathscr{B}_{-k}$.
- If $8 \nmid n$ (with n even) then we have a complete collection of bidihedral braces.
- The subgroup $H = (\langle r \rangle_\bullet, *) = \langle r^2, r \rangle_*$ is dihedral.

Outline

In this section we assume $8 \mid n$.

Fix $k \in K$. Note that k is odd.

Let $d = n/2$, and let $z = r^d$.

Then $Z^\bullet = \langle z \rangle$.

Notice

$$z \circ r^j s^v = r^d \circ r^j s^v = r^{k^v d + j} s^v = z^{k^v} r^j s^v = z r^j s^v$$

and

$$r^j s^v \circ z = r^{j + (-k)^v d} s^v = r^{j+d} s^v = z r^j s^v,$$

so $Z^\circ = \langle z \rangle$ and $z \circ g = zg$ for all $g \in G$. ($Z^\circ = \mathrm{Ann}(\mathscr{A}_k)$.)

Define

$$r^i s^u \blacksquare r^j s^v = (r^i s^u \circ r^j s^v) z^{i(j+v)}.$$

$r^i s^u \blacksquare r^j s^v = (r^i s^u \circ r^j s^v) z^{i(j+v)},\ z = r^d$

Notice $z \in Z^{\blacksquare}$ and $z \blacksquare g = z \circ g = zg$ for all $g \in G$.

If $f = r^i s^u,\ g = r^j s^v,\ h = r^\ell s^w$:

$$\begin{aligned} f \blacksquare (g \blacksquare h) &= f \blacksquare ((g \circ h) z^{j(\ell+w)}) \\ &= (f \circ (g \circ h)) z^{i(j+\ell+v+w)} z^{j(\ell+w)} \\ &= (f \circ g \circ h) z^{ij+i\ell+iv+iw+j\ell+jw} \\ (f \blacksquare g) \blacksquare h &= (f \circ g) z^{i(j+v)} \blacksquare h \\ &= ((f \circ g) \circ h) z^{(i+j)(\ell+w)} z^{i(j+v)} \\ &= (f \circ g \circ h) z^{ij+i\ell+iv+iw+j\ell+jw} \end{aligned}$$

and $G^{\blacksquare}$ is associative.

$g \blacksquare e = e \blacksquare g$ for all $g \in G$.

$r^i \blacksquare (rz)^{-i} = e,\ r^i s \blacksquare r^i s = e.$

$G^{\blacksquare}$ is a group.

...and a brace...

If $f = r^i s^u,\ g = r^j s^v,\ h = r^\ell s^w$:

$$\begin{aligned}(f \blacksquare g) f^{-1} (f \blacksquare g) &= \left((f \circ g) z^{i(j+v)}\right) f^{-1} \left((f \circ h) z^{i(\ell+w)}\right) \\ &= \left((f \circ g) f^{-1} (f \circ h)\right) z^{i(j+v)} z^{i(\ell+w)} \\ &= (f \circ (gh)) z^{i(j+\ell+v+w)} \\ &= f \blacksquare (gh).\end{aligned}$$

Denote the inverse to g in $G^{\blacksquare}$ by $\widetilde{g}$.

Given all of the following:

- $r^{\blacksquare m} = r^m z^{m(m-1)/2}$
- $\widetilde{r} = r^{-1} z$
- $s \blacksquare s = (s \circ s) z^0 = e$
- $s \blacksquare r = (s \circ r) z^0 = r^{-k} s$
- $\widetilde{r} \blacksquare s = r^{-1} z \blacksquare s = (r^{-1} \circ s) z \bullet z^{-1} = r^{-k} s = s \blacksquare r$

we get

$$G^{\blacksquare} = \langle r, s : r^{\blacksquare n} = s^{\blacksquare 2} = r \blacksquare s \blacksquare r \blacksquare s = e \rangle_{\blacksquare} \cong D_n$$

and $(G, \bullet, \blacksquare)$ is bidihedral.

$r^i s^u \blacksquare r^j s^v = (r^i s^u \circ r^j s^v) z^{i(j+v)}, \ z = r^d$

Denote this brace by $\mathscr{C}_k$.

This is a new brace family:

$$\begin{aligned} \mathscr{A}_k : &\quad r \circ r = r^2 \\ \mathscr{B}_k : &\quad r * r = e \\ \mathscr{C}_k : &\quad r \blacksquare r = r^2 z = r^{2+d}. \end{aligned}$$

Furthermore $\mathscr{C}_k$ is not isomorphic to any of the braces constructed so far.

$r^i s^u \blacksquare r^j s^v = (r^i s^u \circ r^j s^v) z^{i(j+v)},\ z = r^d$

Here, we have

$$\gamma_r[r] = rz \qquad \gamma_r[s] = r^{k-1}sz$$
$$\gamma_s[r] = r^k \qquad \gamma_s[s] = s$$

This γ-function was predicted in part 1:

From Part I, and from §1

Since $8 \mid n$ we may take $d = n/2$.

$$\gamma_r[r] = r^{1+d} = rz \qquad \gamma_r[s] = r^{(1+d)k-1}s = r^{k-1}sz$$
$$\gamma_s[r] = r^k \qquad \gamma_s[s] = s$$

Uniqueness of representation?

Question. Suppose $\mathscr{C}_{k'} \cong \mathscr{C}_k$. Must $k' = k$?

Note $r \blacksquare s = r^k sz = r^{k+d} s$.

Let $\alpha \in \mathrm{Aut}(G^\bullet), \alpha(r) = r,\ \alpha(s) = rs$, and as before let

$$g \blacksquare' h = \alpha(\alpha^{-1}(g) \blacksquare \alpha^{-1}(h)).$$

Then, e.g.,

$$r \blacksquare' s = \alpha(r \blacksquare r^{-1} s) = \alpha(r \circ r^{-1} s) = \alpha(r^{k-1} s) = r^k s.$$

Since $r \blacksquare' s = r^{k'} sz$ it follows that $k' = k + d$.

In fact, $\mathscr{C}_{k'} \cong \mathscr{C}_{k+d}$ via α.

$(k')^2 = (k+d)^2 = k^2 + 2kd + d^2 = 1$ since $d = n/2$.

On the other hand, $\beta(r) = r^u,\ \beta(s) = s$ is an automorphism of $\mathscr{C}_k$.

Repeat the $\mathscr{B}_k$ process?

Question. Let $\psi \in \mathrm{Ab}(G^{\blacksquare})$, and let

$$g \star h = g \blacksquare \widetilde{\psi(g)} \blacksquare h \blacksquare \psi(g).$$

Is $(G, \bullet, \star)$ a brace?

Yes, but.

For $\psi_c \in \mathrm{Ab}(G^{\blacksquare})$ we can denote this brace by $\mathscr{D}_{k,c}$.

Then

$$\mathscr{D}_{k,c} = \mathscr{B}_{(-1)^c k, c+d/2},$$

so they're not new.

Properties. (Recall $\kappa_n = |K_n|$.)

- The ideal $H = (\langle r \rangle_\bullet, \blacksquare) = \langle r \rangle_\blacksquare$ is cyclic.
- The ideal $H \subseteq (G, \bullet, \blacksquare)$ is a nontrivial subbrace of $\mathscr{C}_k$.
- $\widehat{\mathscr{C}_k} = \mathscr{C}_{-k}$.
- If $8 \mid n$ then we have all bidihedral braces.

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What's next

$\kappa_n = |K_n|,\ K_n = \{k \in \mathbb{Z}_n : k^2 = 1\}$

Theorem

The number $\beth_n$ of bidihedral braces of order $2n$ is

$$\beth_n = \begin{cases} \kappa_n & n \equiv 1,3,5,7 \pmod 8 \\ 2\kappa_n & n \equiv 2,4,6 \pmod 8 \\ \frac{5}{2}\kappa_n & n \equiv 0 \pmod 8 \end{cases}.$$

Proof.

- n odd. Choice of $k \in K$ gives $\mathscr{A}_k$.
- $n \equiv 2,4,6 \pmod 8$. Choice of $k \in K$ gives $\mathscr{A}_k$ and $\mathscr{B}_k$.
- $n \equiv 0 \pmod 8$. Choice of $k \in K$ gives $\mathscr{A}_k$; $\mathscr{B}_k$; and $\mathscr{C}_k \cong \mathscr{C}_{k+d}$.

□

Outline

Reset notation.

Let L/K be a Galois extension, Galois group G°.

Then each brace $(G, \bullet, \circ)$ corresponds to a Hopf-Galois structure on L/K.

Isomorphic, non-identical braces can give different Hopf-Galois structures.

The number of such structures, $R(D_n, [D_n])$, is enumerated in (Kohl, 2020).

On Kohl's work

The number of such structures, $R(D_n, [D_n])$, is enumerated in (Kohl, 2020).

$$R(D_n, [D_n]) = \begin{cases} \kappa_n & n \equiv 1, 3, 5, 7 \pmod 8 \\ (1+n)\kappa_n & n \equiv 2, 6 \pmod 8 \\ (1+\frac{n}{2})\kappa_n & n \equiv 4 \pmod 8 \\ (2+\frac{n}{2})\kappa_n & n \equiv 0 \pmod 8 \end{cases}.$$

- $n \equiv 1, 3, 5, 7 \pmod 8$. From $\mathscr{A}_k$.
- $n \equiv 2, 6 \pmod 8$. From $\mathscr{A}_k$ and $\mathscr{B}_{k,c}$, $0 \leq c < n$.
- $n \equiv 4 \pmod 8$. From $\mathscr{A}_k$ and $\mathscr{B}_{k,c}$, $0 \leq c < n/2$.
- $n \equiv 0 \pmod 8$ From $\mathscr{A}_k$; $\mathscr{B}_{k,c}$, $0 \leq c < n/2$; $\mathscr{C}_k$.

Outline

Twisting braces

Suppose $(G, \bullet, \circ)$ is a brace, and $\psi \in \mathrm{Ab}(G^\circ)$.

Let

$$g * h = g \circ \overline{\psi(g)} \circ h \circ \psi(g).$$

1. When is $g * h = g \circ (\overline{\psi(g)} h \psi(g))$?
2. When is $(G, \bullet, *)$ a brace?
3. When is $(G, \bullet, *)$ biskew?

Thank you.