Regular Quasigroup Actions and Zappa-Szep Products

Timothy Kohl

Boston University

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Zappa-Szep Products

Definition

A group G is a Zappa-Szep product of two of its proper subgroups $A \leq G$, and $B \leq G$ if $G = AB$ and $A \cap B = \{e_G\}$.

The key observation to make is that, in this decomposition, neither constituent subgroup is normal, indeed, semi-direct products are a special case where either A or B (or both) are normal subgroups of G.

This phenomenon is not that uncommon, but is less well studied since normal structure cannot be used, for example, to put restrictions on the interactions between the two subgroups.

However, there are actually many examples one can find.

For example, $G = S_4$ can be decomposed into over 60 such products, of which over 40 are not semi-direct products, such as:

$$A = \langle (2,4,3) \rangle$$
$$B = \langle (1,4)(2,3),(1,3)(2,4),(3,4) \rangle$$

Other examples include $G = GL_n(\mathbb{C})$ for $A = U(n)$ (unitary group) and $B = \{invertible\ upper\ triangular\ matrices\}$ which arises from the QR decomposition in linear algebra.

And, of fairly central importance in group theory, the Hall decomposition of a group.

Specifically if G is solvable, where $|G| = n$ is divisible by d, where $gcd(n, n/d) = 1$ then G has subgroups A, B where $|A| = d$ and $|B| = n/d$ where $G = AB$ is Zappa-Szep product.

As such this is not a rare phenomenon at all.

Circling back to S_4 for a moment, a more 'obvious' decomposition is this:

$$A = \langle (1,2,3,4) \rangle$$
$$B = \langle (2,3),(2,4) \rangle$$

where, in particular, the subgroup B is a copy of S_3 consisting of all permutations fixing '1', and the subgroup A acts regularly on $\{1,2,3,4\}$.

Let's refresh ourselves on the definition of regularity.

If $|X| = n$ then a subgroup $N \leq B = \text{Perm}(X)$ is **regular** if any two of the following conditions hold:

- N acts transitively on X
- N acts freely on X (i.e. $\sigma(x) = x$ implies $\sigma = e_N$)
- $|N| = n$.

We note also that a subgroup of B is termed **semi-regular** if it acts freely, i.e. fixed point freely, so in particular any subgroup of a regular permutation group is semi-regular.

The relevance of regular permutation groups to Hopf-Galois theory and braces is already familiar to those in the audience, but I am interested in group actions and the relevance to group structure.

Transitive Groups Containing Regular Subgroups

Our starting point are two papers from 1935, one by G.A. Miller [5], and the other by B.H. Neumann [6].

Miller's result is this:

Theorem

If G is a transitive permutation group of finite degree, containing a regular subgroup R, and F is any point-stabilizer subgroup of G then $G = RF$ is a Zappa-Szep product.

The example of the decomposition of S_4 as $\langle(1,2,3,4)\rangle \cdot \langle(2,3),(2,4)\rangle$ is a fairly obvious instance of this decomposition, and indeed generalizes to all S_n.

Moreover, as the conjugate of any point stabilizer is another point stabilizer, and as regularity is preserved by conjugacy, then basically *any point stabilizer will do*.

We note also that in this setup, the stabilizer is therefore obviously not normal, but that it is possible for the regular complement to be normal, which would make it a semi-direct product, but again this is not automatic.

A natural example of this is the holomorph of a group N, given that in $\rho(N)Aut(N)$, one has the regular normal complement $\rho(N)$ to the point stabilizer $Aut(N)$.

In the paper by Neumann, [6], he states a characterization of Zappa-Szep products, which implies that they are *always* representable as a transitive group with a regular subgroup, complementing a stabilizer subgroup.

(Basically, Miller observed the sufficiency, Neumann considers the necessity.)

Namely, any (finite) Zappa-Szep product, $G = AB$ arises from a transitive permutation representation of G where, without loss of generality, A corresponds to the regular component, and B corresponds to the point stabilizer, in the induced transitive action of G.

But what is this action?

The basic idea is that if $G = AB$ then consider the action of G on left cosets of G/B, where the elements of A are a natural choice of left coset representatives.

As such, the elements of A act regularly, whence the induced action from all of G yields a patently transitive action.

However... this is not altogether correct without additional conditions.

If G is to act faithfully on the left coset space G/B then B must be core free, namely $\bigcap_{g\in G} gBg^{-1}$ must be trivial.

But from the view of G as a transitive group acting on a set X, one has that all point stabilizers are conjugate, and so their intersection must be trivial.

So, getting back to an arbitrary decomposition of G as the Zappa-Szep product AB, if B is to be viewed as a point stabilizer, then it must be core-free, otherwise the left action of G is not effective.

If B were not core-free, the left action would still be transitive, but the image would not be isomorphic to G.

Also, for some Zappa-Szep products, there are basic constraints due to order which compel one to view only one of the components as being regular in the induced action.

For example $G = D_4 \times A_4$ (of order 96) can be represented as a Zappa-Szep product $(C_2 \times D_4)C_6$ and there is a transitive representation, in degree 16 where the regular component is $C_2 \times D_4$ and the point stabilizer corresponds to the C_6 component, all within S_{16} of course.

However, there is no transitive subgroup of S_6 of order 96, so the order 6 subgroup of G cannot be represented as a regular subgroup of a transitive action (in degree 6) by a group isomorphic to G.

(So in a way, this indirectly implies that the order 16 component can't possibly be core free.)

Also, one can show that any abelian transitive subgroup of S_n must actually be regular.

So for example (amongst many!) if p is a prime and $G = C_p \times C_p$, then it is (trivially) a Zappa-Szep product, but there is no transitive subgroup of S_p isomorphic to G.

As such, for some Zappa-Szep products, AB, there is *no* transitive action where either A or B can be represented as the regular/stabilizer component.

Nonetheless, these results, Miller's in particular, are still very interesting when they apply, (i.e. when at least one component is core-free) since it shows how the combinatorial/permutational decomposition directly corresponds to the (abstract) decomposition, where the latter would seem to be independent of the nature of any particular representation.

More recent publications such as [4] consider regular subgroups of particular classes of transitive groups, with a view of Zappa-Szep products as the foundational setting.

The earlier examples would also make one think that any transitive subgroup can be decomposed in this fashion, however this is not the case.

Cameron et. al. in [1] (generalizing earlier work [2] of Fein, Kantor, and Schacher from 1981) give a number of examples of what are termed **elusive** permutation groups, namely transitive permutation groups, with no semi-regular (let alone regular) subgroups.

That is, while the stabilizer can of course be defined with respect to whatever transitive action one has, the question is to whether the regular complement exists.

We will use the term elusive to mean having no regular subgroups.

Some of the ones they construct are derived from actions of affine groups over degree 2 extensions of finite fields $\mathbb{F}_p$ where p is a Mersenne prime.

Another they give derives from a certain degree 12 representation of the Mathieu group M_{11}, such as this for example:

$$G = \langle (1,12,11,6,8,5,9,10)(2,3,4,7),(1,7,12,8)(2,5)(3,10,6,11)(4,9)\rangle$$

which is transitive but has no regular subgroups.

The lowest degree examples occur in degree 6, which we will examine shortly.

The question we consider then is, what can we do when the group is elusive, but we would like to represent it as a Zappa-Szep product of some sort?

We pause to point out a curious finding in this regard, namely that one may have two different transitive permutation groups that are isomorphic, where one is elusive, but the other is not.

For example, we have the following transitive representations of A_4 in degrees 6 and 4

$$\langle (1,4)(2,5),(1,3,5)(2,4,6)\rangle \text{ elusive}$$
$$\langle (1,2,3),(2,3,4)\rangle \text{ non-elusive}$$

where the latter corresponds to the Zappa-Szep product $A_4 \cong VC_3$ where $V = \langle (1,3)(2,4),(1,2)(3,4)\rangle$ is Klein-4.

One can even have such a 'disparity' occur in the same degree, such as

$$\langle (1,4)(2,5),(1,3,5)(2,4,6),(1,5)(2,4)\rangle \cong S_4 \text{ elusive}$$
$$\langle (1,4)(2,5),(1,3,5)(2,4,6),(1,5)(2,4)(3,6)\rangle \cong S_4 \text{ non-elusive}$$

where the latter gives rise to the Zappa-Szep product $S_4 \cong S_3 C_4$ for $S_3 = \langle (1,4)(2,3)(5,6),(1,3,5)(2,4,6)\rangle$.

A larger pair of examples in degree 12:

$G = \langle (1,9,5)(2,4,3)(6,8,7)(10,12,11), (1,11,6)(2,9,7)(3,10,5)(4,8,12),$
$(1,7)(2,11)(3,12)(4,10)(5,8)(6,9) \rangle$

$\tilde{G} = \langle (2,8)(3,9)(4,10)(5,11), (1,5,9)(2,6,10)(3,7,11)(4,8,12),$
$(1,12)(2,3)(4,5)(6,7)(8,9)(10,11) \rangle$

Both G and $\tilde{G}$ are isomorphic to the direct product $A_4 \times C_2$, but for G the stabilizer F of 1 is the C_2 direct factor, so that the regular complement is the A_4 direct factor.

And in $\tilde{G}$ the stabilizer of 1 is a copy of C_2 that has no complement, regular or otherwise.

Reality Check

In a way, this idea is not that exotic if one starts with something **really** basic like the orbit stabilizer theorem.

In particular if G acts on a set X then for any $x \in X$, one has $|Gx| = [G : G_x]$ for G_x the point stabilizer, yielding, for transitive actions, that $|G| = |X| \cdot |G_x| = |G/G_x| \cdot |G_x|$.

So the results of Neumann and Miller, can be viewed as a version of this, where now the transitive group G is representable as a Zappa-Szep product of a point stabilizer, i.e. $F = G_x$ and a 'regular' subgroup $R = G/G_x$...?

So *if* $F = G_x$ has a regular complement R in G then the elements of this group are obviously a transversal for G/G_x, where the orbit of this regular complement on 'x' yields all of X by regularity.

But in general of course, if G is elusive then we kind of have a transversal in search of the structure of a regular permutation (sub)-group of G, but of course, in this situation **no** transversal has the structure of a regular subgroup of G.

However, not all transversals are equal.

Semiregular Permutations vs. Derangements

To frame this discussion, we go back to the definition of (semi)regularity and recall that a subgroup $N \leq S_n$ is semiregular if the only element of N with fixed points is the identity.

In particular this implies that if $\sigma \in N$ is non-trivial then it has a cycle decomposition of the form $\sigma = \sigma_1 \cdots \sigma_r$ where all σ_i have the same order, say $|\sigma_i| = k$ and that $rk = n$, and we would call such an element a **semiregular permutation**.

So for example $\sigma = (1,2,3)(4,5,6)(7,8,9)(10,11,12) \in S_{12}$ is a semiregular permutation.

The reason for this is that if a given permutation has no fixed points, no non-trivial power of it can either.

But in contrast

$$\tau = (1,2,3)(4,5,6,7,8,9,10,11,12) \in S_{12}$$

would not be semiregular, but it *does* have the property that every element of $\{1,2,\ldots,12\}$ is moved by τ, so that τ would be an example of a **derangement**, namely an element of S_n that leaves no element fixed.

The point is, while semiregular permutations are derangements, the converse is obviously false, and (recalling the comment about non-trivial powers of a fixed point free element also being fixed point free in a semiregular subgroup) we note that a derangement like τ above cannot lie in any semiregular subgroup of S_{12} as

$$\tau^3 = (4,7,10)(5,8,11)(6,9,12)$$

which would have fixed points.

So if now we have a transitive group $G \leq S_n$ where $F = G_x \leq G$ is a point stabilizer, then any transversal to F in G will have the property that the orbit of x under the action of the elements of this transversal will consist of all of $\{1, \ldots, n\}$.

But as observed earlier, there exists so called **elusive** transitive groups with no regular subgroup, and thus no transversal of F in G will have the structure of a subgroup, let alone a regular subgroup.

However, due to a classical theorem due to Jordan [3], all transitive permutation groups contain at least one derangement, not necessarily semiregular though.

And moreover, by Cameron and Cohen [7], there are at least $\dfrac{(r-1)|G|}{n}$ derangements in any transitive subgroup $G \leq S_n$, where r is the permutation rank of G, which is the number of orbits under $F = G_x$.

So for an elusive permutation group $G \leq S_n$ and $F = G_x$ a point stabilizer, we will look for a transversal R of F in G which (except for the trivial coset representative) consists of derangements.

Now, there are examples of elusive groups G where for a stabilizer F, there does not exist a transversal consisting of derangements, fundamentally because some cosets contain no derangements whatsoever.

For example $G = \langle (1,4)(2,5), (1,3,5)(2,4,6) \rangle \cong A_4$ has $F = G_1 = \langle (2,5)(3,6) \rangle$ where the distinct cosets are

$$
\begin{aligned}
&\{(), (2,5)(3,6)\} \\
&\{(1,4)(3,6), (1,4)(2,5)\} \\
&\{(1,3,5)(2,4,6), (1,6,5)(2,4,3)\} \\
&\{(1,3,2)(4,6,5), (1,6,2)(3,5,4)\} \\
&\{(1,2,3)(4,5,6), (1,5,3)(2,6,4)\} \\
&\{(1,2,6)(3,4,5), (1,5,6)(2,3,4)\}
\end{aligned}
$$

However, computational evidence shows that for many (but not all) elusive permutation groups, there are transversals consisting entirely of derangements.

For example, consider

$$G = \langle (2,4,6),(1,5)(2,4),(1,4,5,2)(3,6) \rangle \cong (C_3 x C_3) \rtimes C_4$$

where if we let $F = G_1$ then $F = \langle (3,5)(4,6),(2,4,6) \rangle \cong S_3$.

One can find a transversal

$$R = \{(),(1,2)(3,4,5,6),(1,3,5)(2,6,4),(1,4,3,6)(2,5),(1,5,3)(2,4,6),(1,6,5,4)(2,3)\}$$

where each non-trivial element is a derangement, and also one can see that the orbit of '1' under the action of the elements of R is $\{1,\ldots,6\}$.

But what does this have to do with regularity, and the theorem about decomposing a transitive group into a product of a regular subgroup and a stabilizer subgroup?

Consider the definition of regularity again.

A subgroup $N \leq S_n$ is regular if any two of the following conditions are satisfied:

- N is transitive on $\{1, \ldots, n\}$
- N acts without fixed points, i.e. for $n \in N$ one has $n(x) = x$ only if n is the identity
- $|N| = n$

So.. if we look at our previous example of the transversal

$$R = \{(), (1,2)(3,4,5,6), (1,3,5)(2,6,4), (1,4,3,6)(2,5), (1,5,3)(2,4,6), (1,6,5,4)(2,3)\}$$

we see that, setwise, this set of elements satisfies all three of these conditions, except of course R is not a subgroup.

Indeed the non-semiregular derangements above have powers which have fixed points of course, but this points to an essential observation, if we're not multiplying the elements of R using the ambient group operation, we would not have this issue.

But then, do we actually have a multiplication at all?...

Quasigroups and Loops

Definition

A set Q, with binary operation '$*$' is

(a) a *left quasigroup* if for each $a, b \in Q$, the equation $a * x = b$ has a unique solution

(b) a *right quasigroup* if for each $a, b \in Q$, the equation $y * a = b$ has a unique solution

(c) a *quasigroup* if it is a left and right quasigroup

(d) a *left loop* if it is a left quasigroup and there exists an element $1 \in Q$ such that $1 * a = a$ for all $a \in Q$

(e) a *right loop* if it is a right quasigroup and there exists an element $1 \in Q$ such that $a * 1 = a$ for all $a \in Q$

(f) a *loop* if it is a left and right loop

For a quasigroup $(Q, *)$ the 'Cayley table' is a latin square, and indeed a latin square on a set yields a quasigroup structure.

What separates quasigroups from semigroups for example, is that semigroups are assumed to be associative whereas quasigroups are not.

Indeed, an associative loop is a group.

The example of most interest to us in this discussion is when we have a subgroup $H \leq G$, and a left transversal $T = \{t_i\}$.

In this case we can define a binary operation $*$ on T by declaring $t_i * t_j = t_k$ where t_k is that left-coset representative such that $t_k H = t_i t_j H$.

It is easily verified that this yields a left quasigroup structure on $(T, *)$ since if $t_i t_j H = t_i t_r H$ then obviously $t_j H = t_r H$.

And if we assume that $e_G \in T$ then we have the structure of a left loop since clearly $e_G * t_j = e_G t_j = t_j$.

Generally speaking though, we don't always get a right quasigroup structure, although our earlier example does actually.

For our earlier example of an elusive group, in degree 6,

$$G = \langle (2,4,6), (1,5)(2,4), (1,4,5,2)(3,6) \rangle$$
$$F = \langle (3,5)(4,6), (2,4,6) \rangle \ i.e. G_1$$
$$R = \{(), (1,2)(3,4,5,6), (1,3,5)(2,6,4), (1,4,3,6)(2,5), (1,5,3)(2,4,6), (1,6,5,4)(2,3)\}$$

if we label the elements of the transversal R as $\{1, \ldots, 6\}$ (conveniently corresponding to the 'orbit' of 1 under the action of R), then the coset multiplication rule previously defined gives the following 'Cayley table'.

*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	5	6	3
3	3	6	5	2	1	4
4	4	5	6	3	2	1
5	5	4	1	6	3	2
6	6	3	2	1	4	5

demonstrating that we have a loop/normalized latin square, (ie. uniqueness in both rows and columns) but alas, typically we only get row uniqueness.

For example, if we change things a bit and consider this elusive group in degree 6:

$$G = \langle (1,4)(2,5), (1,3,5)(2,4,6), (1,5)(2,4) \rangle \cong S_4$$
$$F = \langle (2,3)(5,6), (2,5)(3,6) \rangle \cong C_2 \times C_2 \ i.e.\ G_1$$
$$R = \{(), (1,2,3)(4,5,6), (1,3,5)(2,4,6), (1,4)(2,5), (1,5,3)(2,6,4), (1,6,2)(3,5,4)\}$$

so that R acts regularly on $\{1, \ldots 6\}$ and again, labelling the elements of the transversal R as $\{1, \ldots, 6\}$, we get a table like this:

*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	5	6	4
3	3	4	5	6	1	2
4	4	5	3	1	2	6
5	5	6	1	2	3	4
6	6	1	5	3	4	2

which shows that $(R, *)$ is a left loop, but not a right loop.

Motivation?

My own interest goes back to the notion of the quasiholomorph, $\mathrm{QHol}(G)$, whose construction we recall here:

For G a finite group, with $\mathrm{Hol}(G) = \mathrm{Norm_B}(G)$ for $B = \mathrm{Perm}(G)$ we have the following collections of regular subgroups:

- $\mathcal{S}(G) = \{regular\ N \leq \mathrm{Hol}(G)\ | N \cong G\}$
- $\mathcal{R}(G) = \{regular\ N \leq B \mid N \cong G\ and\ \lambda(G) \leq \mathrm{Norm_B}(N)\}$
- $\mathcal{Q}(G) = \bigcap_{N \in \mathcal{S}(G) \cap \mathcal{R}(G)} \{M \in \mathcal{S}(G) \cap \mathcal{R}(G)\ | N\ normalizes\ M\}$

where the hallmark of $\mathcal{Q}(G)$ is that all its members mutually normalize each other.

(And from these we have a normalizing graph, and each pair of elements gives rise to a bi-skew brace etc.)

Every $N \in \mathcal{Q}(G)$ is a conjugate of $\lambda(G)$ so if we define $\pi(G)$ to be the set of $\alpha_N \in B$ such that $N = \alpha_N \lambda(G) \alpha_N^{-1} \in \mathcal{Q}(G)$, then $|\pi(G)| = |\mathcal{Q}(G)|$ where indeed any element of the coset $\alpha_N \operatorname{Hol}(G)$ conjugates $\lambda(G)$ to N.

So we define $\mathrm{QHol}(G)$, the quasiholomorph of G to be the group generated by

$$\bigcup_{\alpha_N \in \pi(G)} \alpha_N \operatorname{Hol}(G)$$

which, if for example $\mathcal{Q}(G) = \mathcal{S}(G) \cap \mathcal{R}(G)$, is guaranteed to be a group.

Expressing $\mathrm{QHol}(G)$ as a union of cosets, with $\pi(G)$ being patently and overtly a transversal, we typically have that $\mathrm{QHol}(G)$ is typically a Zappa-Szep product, where the transversal $\pi(G)$ is a group (if we sanely decided to let it include e_G etc.) and where $\pi(G)$ acts regularly on $\mathcal{Q}(G)$, with $\operatorname{Hol}(G)$ being the stabilizer of $\lambda(G)$.

However.. there are some G where QHol(G) turns out to be elusive, where, as a cause/consequence, no transversal $\pi(G)$ is a subgroup of QHol(G), and so QHol(G) is *not* a Zappa-Szep product.

However, it still acts transitively on $\mathcal{Q}(G)$.

Among low order groups, the first example where this happens is when $G = P_1$ the Pauli group of order $4^{1+1} = 16$, which I talked about in some detail last year.

Without getting too bogged down in the details, there are many presentations of the Pauli group, one in particular being as a relative holomorph of the quaternion group Q_2, namely

$$Q_2 \rtimes \langle c_i \rangle$$

where $c_i \in Aut(Q_2)$ is conjugation by i.

Note, $\mathrm{QHol}(G)$ is a subgroup of $B = Perm(G)$, but if we want to look at the transitive action of it on $\mathcal{Q}(G)$ we must look at it inside, basically, $Perm(\mathcal{Q}(G))$.

For $G = P_1$ we have $|\mathcal{Q}(G)| = 224$ and what I am interested in, is not only what $\mathrm{QHol}(G)$ looks like as a subgroup of S_{224} but, despite it being elusive, whether there is a transversal of derangements, which can then be construed as yielding a "regular" action by the resulting quasigroup.

Using GAP, we computed this transitive group.

Note, $|\,\mathrm{QHol}(G)| = 224 \cdot |\,\mathrm{Hol}(G)| = 224 \cdot 768 = 172032$ and the image of this inside $Perm(\mathcal{Q}(G)) \cong S_{224}$ should have a smaller stabilizer.

We find that the image Q is isomorphic to

$$(C_2 \times C_2 \times C_2) \rtimes PSL_3(\mathbb{F}_2) \cong \mathbb{F}_2^3 \rtimes GL_3(\mathbb{F}_2) \cong Hol(\mathbb{F}_2^3)$$

where the stabilizer of $\lambda(G)$ is isomorphic to S_3, and that it is elusive.

Computation also shows that one can indeed find a transversal of $F = Q_{\lambda(G)}$ inside Q, consisting of derangements, and this transversal therefore acts regularly, and has the structure of a quasigroup of order 224.

Again, we point out that for most elusive groups, one can find transversals of the stabilizer which, except for the identity, consist of derangements, yielding this type of pseudo-regularity.

Thank You!

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