Isomorphism classes of Galois extensions

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Contents

1. Galois extensions

Throughout this talk, K is a finite extension of $\mathbb{Q}$.

The notion of a Galois extension of K is due to M. Auslander and O. Goldman [AG60].

Let A be a finite dimensional commutative K-algebra and let $\mathrm{End}_K(A)$ denote the K-algebra of K-linear maps $\phi : A \to A$. Let $\mathrm{Aut}_K(A)$ denote the group of K-algebra automorphisms of A.

Let F be a finite subgroup of $\mathrm{Aut}_K(A)$ with $K = A^F$. Let $D(A, F)$ denote the crossed product algebra of A by F.

Then A is an *F-Galois extension of K* if the map

$$j : D(A, F) \to \mathrm{End}_K(A),$$

defined as $j(\sum_{g \in F} a_g g)(t) = \sum_{g \in F} a_g g(t)$, $a_g, t \in A$, is an isomorphism of K-algebras.

The definition of F-Galois extension generalizes the usual definition of a Galois extension of fields.

Let A, A' be F-Galois extensions of K. Then A *is isomorphic to* A' as F-Galois extensions of K if there exists an isomorphism of commutative K-algebras $\theta : A \to A'$ for which

$$\theta(g(x)) = g(\theta(x))$$

for all $g \in F$, $x \in A$.

Let $\mathcal{G}al(K, F)$ denote the set of isomorphism classes of F-Galois extensions of K.

Example 1.

Let $\mathrm{Map}(F, K)$ denote the K-algebra of maps $\phi : F \to K$. Then $\mathrm{Map}(F, K)$ is the *trivial F-Galois extension* of K with action defined as

$$g(\phi)(h) = \phi(g^{-1}h)$$

for $g, h \in F$, $\phi \in \mathrm{Map}(F, K)$.

The set of maps $\{v_g\}_{g \in F}$ defined as

$$v_g(h) = \delta_{g,h},$$

for $g, h \in F$, is a K-basis for $\mathrm{Map}(F, K)$.

$\square$

The Galois extensions are completely determined by the following.

Theorem 2.
Let K be a field, let F be a finite group and let A be an F-Galois extension of K. Then

$$A = \underbrace{L \times L \times \cdots \times L}_{n}$$

where L is a U-Galois field extension of K for some subgroup U of F of index n. (L is a Galois extension of K with group U in the usual sense.)

Proof. See[Pa90, Theorem 4.2].

□

One of the main points of this talk is to answer the following question: Can we recover A from the data U, L? In other words: can A be induced from the data U, L?

The answer is yes (of course), and to do this we need the induction map of Pareigis.

2. The Induction map of Pareigis

Let $U \leq F$ and suppose that $M \in \mathcal{G}al(K, U)$ with $n = [F : U]$.

Let $\{g_1, g_2, \ldots, g_n\}$ be a left transversal for U in F and let $\{g_1U, g_2U, \ldots, g_nU\}$ be the set of left cosets of U in F. We take $g_1 = 1$. Let

$$A = \underbrace{M \times M \times \cdots \times M}_{n}$$

with minimal orthogonal idempotents $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$.

Let $\varsigma : F \to S_n$ be defined as $\varsigma(f)(i) = j$ if $fg_iU = g_jU$.

Define an action of F on A as follows: for $f \in F$,

$$f(\sum_{i=1}^{n} m_i \epsilon_i) = \sum_{i=1}^{n} (g_{\varsigma(f)(i)}^{-1} f g_i)(m_i) \epsilon_{\varsigma(f)(i)},$$

for $m_i \in M$, $1 \leq i \leq n$.

Then A is an F-Galois extension of K [Pa90, Theorem 4.2].

In this way, we define a map

$$A(U, \ ) : \mathcal{G}al(K, U) \to \mathcal{G}al(K, F),$$

$M \mapsto \underbrace{M \times M \times \cdots \times M}_{n}$, which is the *Pareigis induction map*.

Theorem 3.

Let F be a finite group. Let A be an F-Galois extension of K. Then there exists a subgroup $U \leq F$ and a U-Galois extension L of K for which

$$A \cong A(U, L/K),$$

as F-Galois extensions of K, i.e., A can be induced from the data $U, L/K$.

Proof. By Theorem 2,

$$A = \underbrace{L \times L \times \cdots \times L}_{n}, \tag{1}$$

where L is a U-Galois field extension of K for some subgroup U of F of index n.

Now with the data $U, L/K$, we construct an F-Galois extension $A(U, L/K)$ using the induction map of Pareigis:

$$A(U, L/K) = \underbrace{L \times L \times \cdots \times L}_{n}.$$

We then show that $A \cong A(U, L/K)$ as F-Galois extensions of K.

□

Proposition 4.

Let $U, V \leq F$ be distinct subgroups that are isomorphic. Let L/K be a Galois field extension with both U and V as its Galois group. Then $A(U, L/K) \cong A(V, L/K)$ as F-Galois extensions of K if and only if U, V are conjugate.

Proof. Suppose that U, V are conjugate. Then there exists $f \in F$ with $U = fVf^{-1}$. Set $h_1 = f$ and let $\{fV, h_2V, h_3V, \ldots, h_nV\}$ denote the set of distinct left cosets of V in F. The group F acts transitively on $\{fV, h_2V, h_3V, \ldots, h_nV\}$.

For $u \in U$, $u = fvf^{-1}$ for some $v \in V$, thus

$$ufV = fvf^{-1}fV = fvV = fV.$$

Moreover, if $gfV = fV$, then for $v \in V$, $gfv = fv'$ for some $v' \in V$. So, $gf = fv''$, $v'' \in V$, so $g = fv''f^{-1} \in U$.

Thus U stabilizes fV; fV is the "base point."

Consequently, there is an isomorphism of F-sets

$$\alpha : \{fV, h_2V, \ldots, h_nV\} \to \{g_1U, g_2U, \ldots, g_nU\},$$

defined as $\alpha(h_iV) = g_iU$, where $g_i \in F$ satisfies $g_ifV = h_iV$.

Next, we define a map

$$\theta : A(V, L/K) \to A(U, L/K)$$

by the rule

$$\theta(\sum_{i=1}^{n} l_i \epsilon_i) = \sum_{i=1}^{n} g_i^{-1}(l_i)\alpha(\epsilon_i),$$

for $l_i \in L/K$, $1 \leq i \leq n$.

Now, for $f \in F$,

$$
\begin{aligned}
f(\theta(\sum_{i=1}^{n} l_i\epsilon_i)) &= f(\sum_{i=1}^{n} g_i^{-1}(l_i)\alpha(\epsilon_i)) \\
&= \sum_{i=1}^{n}(g_j^{-1}fg_i)(g_i^{-1}(l_i))\alpha(\epsilon_j) \\
&= \sum_{i=1}^{n} g_j^{-1}(f(l_i))\alpha(\epsilon_j) \\
&= \theta(\sum_{i=1}^{n} f(l_i)\epsilon_j) \\
&= \theta(f(\sum_{i=1}^{n} l_i\epsilon_i)).
\end{aligned}
$$

Thus $A(U, L/K) \cong A(V, L/K)$ as F-Galois extensions of K.

For the converse, assume that $A(U, L/K) \cong A(V, L/K)$ as F-Galois extensions of K. Then there exists an F-invariant isomorphism of F-sets

$$\alpha : \{U, g_2U, \ldots, g_nU\} \to \{V, h_2V, \ldots, h_nV\}.$$

We have $\alpha(U) = hV$ for some $h \in F$ and so, $uhV = hV$ for all $u \in U$. Let $u \in U$, $v \in V$. Then

$$h^{-1}uhv = v'$$

for some $v' \in V$.

Thus $h^{-1}uh = v'v^{-1} \in V$. If u, u' are distinct elements of U, then $h^{-1}uh$ and $h^{-1}u'h$ are distinct elements of V. Since $|U| = |V|$, we conclude that $h^{-1}Uh = V$, i.e., U, V are conjugate.

$\square$

Example 5. Let $F = S_3$ with presentation

$$S_3 = \langle a, b : \ a^3 = b^2 = 1, ba = a^2 b \rangle.$$

There are 3 isomorphic subgroups of S_3 of order 2:

$$U_1 = \{1, b\}, \quad U_2 = \{1, ba\}, \quad U_3 = \{1, ab\}.$$

Consider the Galois extension $\mathbb{Q}(\sqrt{2})$ of $\mathbb{Q}$, and consider each U_i. $1 \leq i \leq 3$, to be its Galois group.

Since $[F : U_i] = 3$ for $1 \leq i \leq 3$, each $A(U_i, \mathbb{Q}(\sqrt{2})/\mathbb{Q})$ appears as

$$\mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2}),$$

and thus the $A(U_i, \mathbb{Q}(\sqrt{2})/\mathbb{Q})$ are pairwise isomorphic as $\mathbb{Q}$-algebras.

The subgroups U_i are pairwise conjugate. Thus by Proposition 4, $A(U_i, \mathbb{Q}(\sqrt{2})/\mathbb{Q})$ are pairwise isomorphic as S_3-Galois extensions of $\mathbb{Q}$.

$\square$

3. The Haggenmüller-Pareigis bijection

Let N be a finite group with automorphism group $F = \mathrm{Aut}(N)$.

Let $K[N]$ denote the group ring K-Hopf algebra.

Let $\mathcal{F}orm(K[N])$ denote the collection of the isomorphism classes of the forms of $K[N]$.

Theorem 6 (Haggenmüller and Pareigis).

There exists a bijection

$$\Theta : \mathcal{G}al(K, F) \to \mathcal{F}orm(K[N]),$$

which gives a 1-1 *correspondence between the isomorphism classes of* F*-Galois extensions of* K *and forms of* $K[N]$*, where* $F = \mathrm{Aut}(N)$*. For an* F*-Galois extension* A *of* K*, the map* Θ *is given explicitly as the fixed ring*

$$\Theta(A) = (A[N])^F,$$

where F *acts on* A *through the Galois action and on* N *as automorphisms. The fixed ring* $(A[N])^F$ *is an* A*-form of* $K[N]$ *and so belongs to* $\mathcal{F}orm(K[N])$*.*

Proof. See [HP86, Theorem 5].

□

Example 7. Let $N = C_2 \times C_2$. Then $F = \mathrm{Aut}(C_2 \times C_2) = S_3$.

In Example 5, we constructed a collection of S_3-Galois extensions of $\mathbb{Q}$:

$$A(U_i, \mathbb{Q}(\sqrt{2})/\mathbb{Q}),$$

$1 \leq i \leq 3$. The $A(U_i, \mathbb{Q}(\sqrt{2})/\mathbb{Q})$ are pairwise isomorphic as S_3-Galois extensions.

Let

$$\Theta : \mathcal{G}al(\mathbb{Q}, S_3) \to \mathcal{F}orm(\mathbb{Q}[C_2 \times C_2])$$

be the Haggenmüller-Pareigis bijection.

Now, the set of images

$$\Theta(A(U_i, \mathbb{Q}(\sqrt{2})/\mathbb{Q})) = A(U_i, \mathbb{Q}(\sqrt{2})/\mathbb{Q})[C_2 \times C_2]^{S_3}$$

are pairwise isomorphic as $\mathbb{Q}$-Hopf algebras (and as Hopf forms of $\mathbb{Q}[C_2 \times C_2]$).

$\square$

4. Hopf-Galois structures

By Greither/Pareigis, certain Hopf forms of $K[N]$ arise as Hopf-Galois structures on Galois field extensions.

Let E/K be a Galois extension with group G, and let $\lambda : G \to \mathrm{Perm}(G)$ denote the left regular representation.

Let $(H_N, \cdot_N)$ be a Hopf-Galois structure on E/K corresponding to a regular subgroup $N \leq \mathrm{Perm}(G)$ that is normalized by $\lambda(G)$.

Necessarily, $|N| = |G|$, yet we may have $N \ncong G$ as groups; the isomorphism class of N is the *type* of $(H_N, \cdot_N)$.

Let $(H_{N'}, \cdot_{N'})$ be some other Hopf-Galois structure on E/K corresponding to a regular subgroup $N' \leq \mathrm{Perm}(G)$ normalized by $\lambda(G)$. Then $(H_{N'}, \cdot_{N'})$ is of the *same type* as $(H_N, \cdot_N)$ if $N' \cong N$.

The K-Hopf algebra H_N of the Hopf-Galois structure is a Hopf-algebra E-form of $K[N]$, which by Galois descent corresponds to a homomorphism

$$\varrho_N : G \to F_N$$

in

$$\mathrm{H}^1(G, \mathbf{Aut}(K[N])(E)) = \mathrm{H}^1(G, F_N),$$

where $F_N = \mathrm{Aut}(N)$.

The homomorphism ϱ_N is given as conjugation by elements of $\lambda(G)$.

The kernel of ϱ_N is a normal subgroup of G given as

$$G_0(N) = \{g \in G : \lambda(g)\eta\lambda(g^{-1}) = \eta, \forall \eta \in N\}.$$

The quotient group $G/G_0(N)$ is isomorphic to a subgroup U_N of F_N, which is the image of ϱ_N.

Let $E_0(N) = E^{G_0(N)}$. Then $E_0(N)$ is Galois extension of K with group U_N. So the Hopf algebra form H_N of $K[N]$ determines the data $U_N, E_0(N)/K$.

Using the Pareigis induction map

$$A(U_N, \ ) : \mathcal{G}al(K, U_N) \to \mathcal{G}al(K, F_N),$$

we compute $A(U_N, E_0(N)/K)$, which is an F_N-Galois extension of K.

Theorem 8 (Kohl and U).

Let E/K be a Galois extension with group G and let $(H_N, \cdot_N)$ be a Hopf-Galois structure on E/K of type N. Let

$$\Theta_N : \mathcal{G}al(K, F_N) \to \mathcal{F}orm(K[N])$$

be the Haggenmüller-Pareigis bijection. Then $A(U_N, E_0(N)/K)$ is the preimage of H_N under Θ_N, i.e., $\Theta_N(A(U_N, E_0(N)/K)) = H_N$.

Proof. See [Theorem 4.3, KU25].

□

Example 9. Let $L/\mathbb{Q}$ be the splitting field of the irreducible quartic $p(x) = x^4 - 10x^2 + 1$, that is, $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

Then $L/\mathbb{Q}$ is Galois with group $C_2 \times C_2 = \{1, \sigma, \tau, \tau\sigma\}$, with Galois action

$$\sigma(\sqrt{2} + \sqrt{3}) = \sqrt{2} - \sqrt{3}, \quad \tau(\sqrt{2} + \sqrt{3}) = -\sqrt{2} + \sqrt{3}.$$

By [By96], $L/\mathbb{Q}$ admits 3 Hopf-Galois structures of type C_4.

They correspond to regular subgroups N_1, N_2, N_3 of $\mathrm{Perm}(G)$ normalized by $\lambda(G)$.

We first consider N_1. As one can check

$$\begin{aligned} G_0(N_1) &= \{g \in C_2 \times C_2 : \lambda(g)\eta\lambda(g^{-1}) = \eta, \forall \eta \in N_1\} \\ &= \{1, \sigma\}. \end{aligned}$$

The fixed field of $\{1, \sigma\}$ is $\mathbb{Q}(\sqrt{2})$, and so,

$$E_0(N_1) = \mathbb{Q}(\sqrt{2}).$$

We have

$$U_{N_1} \cong (C_2 \times C_2)/G_0(N_1) = \{1, a\},\ a^2 = 1,$$

which is the Galois group of $E_0(N_1) = \mathbb{Q}(\sqrt{2})$.

We have $F_{N_1} = \mathrm{Aut}(N_1) = C_2$. So in this case, the subgroup $U_{N_1} = \{1, a\}$ of F_{N_1} is all of F_{N_1}, $U_{N_1} = F_{N_1}$.

The induced F_{N_1}-Galois extension of $\mathbb{Q}$ is

$$A(\{1, a\}, \mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \mathbb{Q}(\sqrt{2}),$$

which is the preimage of H_{N_1} under the Haggenmüller-Pareigis bijection

$$\Theta_{N_1} : \mathcal{G}al(\mathbb{Q}, F_{N_1}) \to \mathcal{F}orm(\mathbb{Q}[N_1]).$$

We next consider N_2. Let H_{N_2} be the $\mathbb{Q}$-Hopf algebra attached to the Hopf-Galois structure corresponding to N_2.

Now, H_{N_2} is a form of $\mathbb{Q}[N_2]$, but since N_1 and N_2 are of the same type (C_4), H_{N_2} is also a form of $\mathbb{Q}[N_1]$, through the isomorphism

$$\psi : N_2 \to N_1.$$

Let $F_{N_2} = \mathrm{Aut}(N_2)$. Then there is an induced isomorphism

$$\hat{\psi} : F_{N_2} \to F_{N_1}$$

defined as $\hat{\psi}(f)(\eta) = (\psi f \psi^{-1})(\eta)$, for $f \in F_{N_2}$, $\eta \in N_1$.

The composition

$$\hat{\psi}\varrho_{N_2} : G \to F_{N_2} \to F_{N_1}$$

is the homomorphism in $\mathrm{H}^1(G, F_{N_1})$ corresponding to the form H_{N_2} of $\mathbb{Q}[N_1]$.

The kernel of $\hat{\psi}\varrho_{N_2}$ is

$$\begin{aligned} G_0(N_2) &= \{g \in C_2 \times C_2 : \lambda(g)\eta\lambda(g^{-1}) = \eta, \forall \eta \in N_2\} \\ &= \{1, \tau\}. \end{aligned}$$

The fixed field of $\{1, \tau\}$ is $\mathbb{Q}(\sqrt{3})$, and so,

$$E_0(N_2) = \mathbb{Q}(\sqrt{3}).$$

We have

$$U_{N_2} \cong (C_2 \times C_2)/G_0(N_2) = \{1, b\},\ b^2 = 1$$

which is the Galois group of $E_0(N_2) = \mathbb{Q}(\sqrt{3})$, *as a subgroup of* F_{N_2}.

As a subgroup of F_{N_1}, the Galois group of $\mathbb{Q}(\sqrt{3})$ is

$$\hat{\psi}(U_{N_2}) = F_{N_1} = U_{N_1} = \{1, a\}.$$

Note: U_{N_1} acts on $\mathbb{Q}(\sqrt{3})$ through $\hat{\psi}^{-1}$, i.e.,

$$a(\sqrt{3}) = \hat{\psi}^{-1}(a)(\sqrt{3}) = b(\sqrt{3}) = -\sqrt{3},$$

as expected.

With the data $\{1, a\}$, $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$, the induced F_{N_1}-Galois extension of $\mathbb{Q}$ is

$$A(\{1, a\}, \mathbb{Q}(\sqrt{3})/\mathbb{Q}) = \mathbb{Q}(\sqrt{3}),$$

and we have

$$\Theta_{N_1}(\mathbb{Q}(\sqrt{3})) = H_{N_2}.$$

Finally, we consider N_3. Let H_{N_3} be the $\mathbb{Q}$-Hopf algebra attached to the Hopf-Galois structure corresponding to N_3.

Now, H_{N_3} is a form of $\mathbb{Q}[N_3]$, but since N_1 and N_3 are of the same type (C_4), H_{N_3} is also a form of $\mathbb{Q}[N_1]$, through an isomorphism

$$\psi : N_3 \to N_1.$$

Let $F_{N_3} = \mathrm{Aut}(N_3)$. Then, as before, there is an induced isomorphism

$$\hat{\psi} : F_{N_3} \to F_{N_1}.$$

The composition

$$\hat{\psi}\varrho_{N_3} : G \to F_{N_3} \to F_{N_1}$$

is the homomorphism in $\mathrm{H}^1(G, F_{N_1})$ corresponding to the form H_{N_3} of $\mathbb{Q}[N_1]$.

The kernel of $\hat{\psi}\varrho_{N_3}$ is

$$\begin{aligned} G_0(N_3) &= \{g \in C_2 \times C_2 : \lambda(g)\eta\lambda(g^{-1}) = \eta, \forall \eta \in N_3\} \\ &= \{1, \tau\sigma\}. \end{aligned}$$

The fixed field of $\{1, \tau\sigma\}$ is $\mathbb{Q}(\sqrt{6})$, and so,

$$E_0(N_3) = \mathbb{Q}(\sqrt{6}).$$

We have

$$U_{N_3} \cong (C_2 \times C_2)/G_0(N_3) = \{1, c\},\ c^2 = 1,$$

which is the Galois group of $E_0(N_3) = \mathbb{Q}(\sqrt{6})$, *as a subgroup of* F_{N_3}.

As a subgroup of F_{N_1}, the Galois group of $\mathbb{Q}(\sqrt{6})$ is

$$\hat{\psi}(U_{N_3}) = F_{N_1} = U_{N_1} = \{1, a\}.$$

Note: U_{N_1} acts on $\mathbb{Q}(\sqrt{6})$ through $\hat{\psi}^{-1}$, i.e.,

$$a(\sqrt{6}) = \hat{\psi}^{-1}(a)(\sqrt{6}) = c(\sqrt{6}) = -\sqrt{6},$$

as expected.

With the data $\{1, a\}$, $\mathbb{Q}(\sqrt{6})/\mathbb{Q}$, the induced F_{N_1}-Galois extension of $\mathbb{Q}$ is

$$A(\{1, a\}, \mathbb{Q}(\sqrt{6})/\mathbb{Q}) = \mathbb{Q}(\sqrt{6}),$$

and we have

$$\Theta_{N_1}(\mathbb{Q}(\sqrt{6})) = H_{N_3}.$$

From the definition of Θ_{N_1}, we obtain

$$H_{N_1} = \mathbb{Q}(\sqrt{2})[N_1]^{\{1,a\}}$$

$$H_{N_2} = \mathbb{Q}(\sqrt{3})[N_1]^{\{1,a\}}$$

$$H_{N_3} = \mathbb{Q}(\sqrt{6})[N_1]^{\{1,a\}}$$

The H_{N_i}, $1 \leq i \leq 3$, are pairwise non-isomorphic as $\mathbb{Q}$-Hopf algebras since they are associated to the preimages in $\mathcal{G}al(\mathbb{Q}, F_{N_1})$: $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$, $\mathbb{Q}(\sqrt{6})/\mathbb{Q}$, respectively, which are clearly pairwise non-isomorphic as F_{N_1}-Galois extensions of $\mathbb{Q}$.

$\square$

5. Another Example

Taylor and Truman [TT19] provide some additional examples.

Let L/K be a Galois extension of fields with quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Taylor and Truman have enumerated the Hopf-Galois structures on L/K of each possible type [Table 1, TT19].

Example 10. [Lemma 3.1, TT19]. There are 2 Hopf-Galois structures of type $C_2 \times C_2 \times C_2$.

Note that $\mathrm{Aut}(C_2 \times C_2 \times C_2) = \mathrm{GL}_3(\mathbb{Z}_2)$, a group of order 168.

Let M_1, M_2 be the regular subgroups of $\mathrm{Perm}(Q_8)$ nomalized by $\lambda(Q_8)$. Let H_{M_1}, H_{M_2} be the associated K-Hopf algebras.

We have

$$G_0(M_1) = G_0(M_2) = \{1, -1\}$$

and

$$E_0(M_1) = E_0(M_2) = K(\alpha, \beta),$$

the unique biquadratic subfield of L/K with Galois group

$$U_{M_1} = U_{M_2} = Q_8/\{1, -1\} = C_2 \times C_2.$$

Let $\varrho_{M_1} : Q_8 \to F_{M_1}$ be the homomorphism corresponding to the L-form H_{M_1} of $K[M_1]$. The kernel of ϱ is $G_0(M_1)$; its image is U_{M_1}. We have

$$A(U_{M_1}, K(\alpha, \beta)/K) = \underbrace{K(\alpha, \beta) \times K(\alpha, \beta) \times \cdots \times K(\alpha, \beta)}_{42}$$

$$\Theta_{M_1}(A(U_{M_1}, K(\alpha, \beta)/K)) = H_{M_1}.$$

Considering the other regular subgroup M_2, there exists a Q_8-invariant isomorphism $\psi : M_2 \to M_1$, inducing the isomorphism

$$\hat{\psi} : F_{M_2} \to F_{M_1}.$$

Since ψ is Q_8-invariant, $\hat{\psi}$ restricted to $U_{M_2} \leq F_{M_2}$ is the identity map.

The homomorphism corresponding to the L-form H_{M_2} of $K[M_1]$ is

$$\hat{\psi}\varrho_{M_2} : Q_8 \to F_{M_2} \to F_{M_1}.$$

The data for H_{M_2} is U_{M_1}, $K(\alpha, \beta)/K$, so in fact,

$$\Theta_{M_1}(A(U_{M_1}, K(\alpha, \beta)/K)) = H_{M_2}.$$

Consequently, $H_{M_1} \cong H_{M_2}$ as K-Hopf algebras, as already shown in [TT19].

$\square$

Thank you!

References

[AG60] Auslander, M.; Goldman, O. The Brauer group of a commutative ring. *Trans. Amer. Math. Soc.* **97** no. 3 (1960), 367–409.

[By96] Byott, N. Uniqueness of Hopf-Galois structures for separable field extensions. *Comm. Algebra*. **24**, (1996), 3217-3228.

[CHR65] Chase, S. U.; Harrison, D. K.; Rosenberg, A. Galois theory and Galois cohomology of commutative rings. *Mem. Amer. Math. Soc.*, no. 52, (1965), 15–33.

[HP86] Haggenmüller, R.; Pareigis, B. Hopf algebra forms of the multiplicative group and other groups. *manuscripta math.* **55** no. 2 (1986), 121–136.

[Ha65] Harrison, D. K. Abelian extensions of commutative rings. *Mem. Amer. Math. Soc.*, no. 52, (1965), 1–14.

[KKTU19b] Koch, A.; Kohl, T.; Truman, P. J.; Underwood, R. Isomorphism problems for Hopf-Galois structures on separable field extensions. *J. Pure and Appl. Algebra* **223** no. 5 (2019), 2230–2245.

[KU25] Kohl, T.; Underwood, R. Galois extensions and Hopf-Galois structures. *New York J. Math.* **31** (2025), 238-258.

[Pa90] Pareigis, B. Forms of Hopf algebras and Galois theory. *Topics in algebra, Part 1 (Warsaw, 1988)*, 75–93, Banach Center Publ., **26**, Part 1, *PWN, Warsaw*, 1990.

[TT19] Taylor, S.; Truman, P. J. The Structure of Hopf algebras giving Hopf-Galois structures on quaternionic extensions. *New York J. Math.*, **25**, (2019), 219-237.